ANNALS OF MATHEMATICS
Vol. 39, No. 4, October, 1938

ON INTERPOLATION II

ON THE DISTRIBUTION OF THE FUNDAMENTAL POINTS OF LAGRANGE AND HERMITE INTERPOLATION[1]

BY P. ERDÖS AND P. TURÁN

(Received November 1, 1937)

1. Introduction.

Let

$$A \equiv \left\{ \begin{array}{l} x_1^{(1)} \\ x_1^{(2)},\ x_2^{(2)} \\ \vdots \\ x_1^{(n)},\ x_2^{(n)},\ \cdots\ x_n^{(n)} \\ \vdots\quad\ \vdots\qquad\ \vdots\ \ddots \\ \vdots\quad\ \vdots\qquad\ \vdots\ \ddots \end{array} \right\}$$

be a triangular matrix, or shortly matrix, where for every line

$$-1 \leqq x_n^{(n)} < x_{n-1}^{(n)} < \cdots < x_1^{(n)} \leqq +1.$$

We define the n^{th} Lagrange interpolation parabola belonging to the function $f(x)$ as the polynomial $L_n(f)$ of degree $(n-1)$ at most taking at $x_1^{(n)}, \cdots, x_n^{(n)}$ the values $f(x_1^{(n)}), \cdots f(x_n^{(n)})$. The explicit form of this polynomial is

$$L_n(f) \equiv \sum_{\nu=1}^{n} f(x_\nu^{(n)}) l_\nu^{(n)}(x) \equiv \sum_{\nu=1}^{n} f(x_\nu) l_\nu(x),$$

where

$$l_\nu(x) = \frac{\omega(x)}{\omega'(x_\nu)(x - x_\nu)},$$

and

$$\omega(x) \equiv \omega_n(x) = \prod_{\nu=1}^{n} (x - x_\nu^{(n)}) \equiv \prod_{\nu=1}^{n} (x - x_\nu).$$

The polynomials $l_\nu(x)$, the "fundamental functions" are independent of $f(x)$. We give explicit indication of the dependence of $l_\nu(x)$ upon n only when we want to emphasize this dependence or when a misunderstanding may arise.

[1] We reported part of these results to the Math. and Phys. Assoc. Budapest, 12. XII. 1935.

We evidently have

$$L_n(1) \equiv \sum_{\nu=1}^{n} l_\nu(x) \equiv 1, \tag{1a}$$

or more generally, if $\psi(x)$ is a polynomial of degree k

$$L_{n+k}(\psi) = \psi \qquad n = 1, 2, \cdots. \tag{1b}$$

In the theory of the Lagrange-interpolation we shall consider the two sequences $L_n(f)_{x=x_0}$ and $\int_{-1}^{1} L_n(f)\,dx$ for $n \to \infty$. The behavior of the first sequence is determined[2] by

$$B_1(n, x_0) = \sum_{k=1}^{n} |\, l_k(x_0)\,|, \tag{2a}$$

that of the second one[3] by

$$B_2(n) = \sum_{\nu=1}^{n} |\, \lambda_\nu^{(n)}\,| \equiv \sum_{\nu=1}^{n} \left| \int_{-1}^{1} l_\nu(x)\,dx \right|. \tag{2b}$$

These expressions $B_1(n, x_0)$ and $B_2(n)$ are evidently independent of the function $f(x)$; they depend only upon the matrix A of the fundamental points and (as in (2a)) upon the value of x_0; they are the analogues of the Lebesgue-constants in the theory of Fourier-series.

The examination of the second problem is particularly easy when the so called "Cotes numbers" are all greater than or equal to 0,[4] i.e.

$$\lambda_k = \lambda_k^{(n)} \geqq 0, \qquad k = 1, 2, \cdots n, \quad n = 1, 2, \cdots. \tag{3}$$

For in this case, by (1a) we have

$$\sum_{k=1}^{n} |\, \lambda_k\,| = \sum_{k=1}^{n} \lambda_k = 2,$$

i.e. we immediately obtain by Pólya (l.c.) that if $f(x)$ is R.-integrable,[2] then for the Lagrange parabolas taken on such an A

$$\lim_{n\to\infty} \int_{-1}^{1} L_n(f)\,dx = \int_{-1}^{1} f(x)\,dx.$$

Thus (3) implies an important interpolation property of the matrix A. Pólya proved that the necessary and sufficient condition for quadrature convergence for continuous functions is

$$B_2(n) < c_1 \tag{4}$$

[2] H. Hahn: *Über das Interpolationsproblem*, Math. Zeitschrift 1918. The notations "bounded," "R-integrable" "L-integrable" "continuous" refer always, to the interval $[-1, +1]$.

[3] G. Pólya, *Über die Konvergenz von Quadraturverfahren*, Math. Zeitschrift 1933. Bd. 37. pp. 264–286.

[4] L. Fejér: *Mechanische Quadraturen mit positiven Cotes-Zahlen*, Math. Zeitschrift 1933. Bd. 37. p. 287–309. See (2b).

where c_1 (and later $c_2 \cdots$) are positive constants independent of n. Also (4) depends only upon A and implies an important interpolation property.

To obtain a new and important interpolation property of A in the theory of the *Lagrange* interpolation we are forced according to Fejér[5] to consider the *Hermite*-interpolation. The n^{th} step parabola of the bounded and integrable function $f(x)$ is defined as the polynomial $H_n(f)$ of degree $(2n-1)$ at most taking at $x_1^{(n)}, \cdots, x_n^{(n)}$ the values $f(x_1^{(n)}), \cdots, f(x_n^{(n)})$ with $\left.\frac{dH_n(f)}{dx}\right|_{x=x_\nu^{(n)}} = 0$. $(\nu = 1, 2, \cdots n.)$ The explicit form of this polynomial is given by

$$H_n(f) = \sum_{\nu=1}^{n} f(x_\nu)\left[1 - \frac{\omega''(x_\nu)}{\omega'(x_\nu)}(x - x_\nu)\right] l_\nu(x)^2 \equiv \sum_{\nu=1}^{n} f(x_\nu) h_\nu(x), \tag{5a}$$

where

$$\omega(x) = \prod_{\nu=1}^{n}(x - x_\nu). \tag{5b}$$

Then the above mentioned property is

$$\theta_k(x) \equiv 1 - \frac{\omega''(x_k)}{\omega'(x_k)}(x - x_k) \geqq c_2, \tag{5c}$$

$$-1 \leqq x \leqq +1, \qquad k = 1, 2, \cdots, n, \quad n = 1, 2, \cdots$$

The matrices with the property (5c) are called by Fejér "strongly normal" matrices and he deduces for their Lagrange parabolas convergence criteria of great generality. The identity $\sum_{k=1}^{n} h_k(x) \equiv 1$ plays an important rôle here. By this identity and (5c) we have for strongly normal matrices $\sum_{k=1}^{n} l_k(x)^2 \leqq \frac{1}{c_2}$ i.e. a fortiori

$$|\, l_k(x)\,| \leqq c_2^{-\frac{1}{2}} \tag{6}$$

$$-1 \leqq x \leqq +1, \qquad k = 1, 2, \cdots n, \quad n = 1, 2, \cdots.$$

Thus the strongly normal matrices satisfy (6), but the converse is not true. (6) implies an important interpolation property, too.

The importance of the Hermite interpolation is also shown by the following fact. As Bernstein[6] proved, there exists for every matrix A a continuous $f(x)$ and an abscissa x_0 such, that $\limsup_{n\to\infty} |\, L_n(f)\,|_{x=x_0} = +\infty$. On the other hand Fejér[7] proved, that for certain special matrices the Hermite parabolas $H_n(f)$ of

---

[5] L. Fejér, *On the Characterization of some remarkable systems, etc.* Amer. Math. Monthly. 1934.

[6] Bernstein, *Sur la limitation des valeurs d'un polynom etc.* Bull. de l'Acad. de Sciences de l'URSS, 1931.

[7] Fejér, *Die Abschätzung ein Polynoms* etc. Math. Zeitsch. 1930, Bd. 32, pp. 426–457.

any continuous $f(x)$ converge uniformly to $f(x)$ in $[-1, +1]$; e.g. the "matrix T", the n^{th} row of which consists of the n roots of $T_n(x)$, the Tschebyscheff polynomial ($T_n(\cos\theta) = \cos n\theta$), displays this property. The question now arises, which matrices possess this property? Or if uniformity of convergence is not required: what is the necessary and sufficient condition, that for *any* continuous $f(x)$ and at any fixed point x_0

$$\lim_{n\to\infty} H_n(f)_{x=x_0} = f(x_0)? \tag{7}$$

For our purpose it will be sufficient to know, that a necessary condition for (7) is

$$\sum_{k=1}^{n} |h_k(x)| \leqq c_3, \qquad -1 \leqq x \leqq 1, \qquad n = 1, 2, \cdots. \tag{8}$$

This condition follows immediately from the theorem of Hahn (l.c.). The sum in (8) evidently depends only upon A; thus it expresses an interesting interpolation property.

In (3), (4), (5c), (6) and (8) we enumerated some interpolation properties. As far as we know, the whole literature on interpolation[8]—with the exception of two papers—is deducing convergence—and divergence—properties from given suppositions for the matrices. Fejér[9] was the first to invert the problem, deducing distribution-properties from given interpolation properties. He proved e.g. that from (3) or from (5c) it follows, that for $n \to \infty$ the difference of the consecutive elements of the n^{th} row of the matrix tends to 0. The importance of the new idea is shown by the fact, that the required interpolation properties are sometimes quite easily verified. An interesting example is given by the "matrix P", the n^{th} row of which is given by the n roots of the n^{th} Legendre polynomial $P_n(x)$. In consequence of the orthogonality we evidently have

$$\int_{-1}^{1} l_\nu(x)\,dx = \int_{-1}^{1} l_\nu(x)^2\,dx > 0, \qquad \nu = 1, 2, \cdots, n, \quad n = 1, 2, \cdots$$

which means that the matrix P satisfies (3).

In this paper we are concerned with analogous investigations; we deduce the distribution of the fundamental-abscissas from given interpolation properties. In §2 we show the effect of the condition (6).

THEOREM I. *Let*

$$x_\nu^{(n)} = \cos\theta_\nu^{(n)}, \qquad 0 = \theta_0^{(n)} \leqq \theta_1^{(n)} < \theta_2^{(n)} < \cdots < \theta_n^{(n)} \leqq \theta_{n+1}^{(n)} = \pi. \tag{9}$$

---

[8] As a matter of fact, we do not mean here complex interpolation.

[9] See footnote 4 and Fejér: *Lagrangesche Interpolation* etc., Math. Ann. (1932).

Then, if matrix A satisfies (6), *we have*

$$\frac{c_4}{n} \leqq \theta_{\nu+1}^{(n)} - \theta_\nu^{(n)} \leqq \frac{c_5}{n}, \qquad \nu = 1, 2, \cdots (n-1) \tag{10a}$$

The upper bound is valid for $\nu = 0$ *and* $\nu = n$, *the lower one is not.*[10]

The theorem generalizes Fejér's second result in two respects; the assumption is weaker and the result stronger. The theorem means, that the distribution of the roots on the circle with the radius 1 is quasi-uniform i.e.

$$\text{(10b)} \quad \begin{cases} \dfrac{c_4}{n} < |\operatorname{arc} Q_\nu^{(n)} Q_{\nu+1}^{(n)}| < \dfrac{c_5}{n}, \qquad \nu = 1, 2, \cdots (n-1) \\ \text{and} \\ |\operatorname{arc} Q_0^{(n)} Q_1^{(n)}| \leqq \dfrac{c_5}{n}, \qquad |\operatorname{arc} Q_n^{(n)} Q_{n+1}^{(n)}| \leqq \dfrac{c_5}{n}. \end{cases}$$

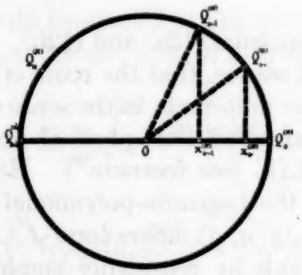


FIG. 1

As stated above, from theorem I it follows a fortiori, that the distribution of the fundamental points Fig. 1. of a strongly normal matrix is quasi-uniform.[11]

There is an application of theorem I for the roots of some classical polynomials. The Jacobi-polynomials $J_n(x, \alpha, \beta)$ corresponding to the parameters α, β ($\alpha \geqq 0$, $\beta \geqq 0$, α and β fixed, $n = 1, 2, \cdots$) may be characterised as the polynomial solutions of the differential equation

$$(1 - x^2)\frac{d^2 J_n}{dx^2} + 2[(\alpha - \beta) - (\alpha + \beta)x]\frac{dJ_n}{dx} + n[n + 2(\alpha + \beta) - 1]J_n = 0. \tag{11}$$

---

[10] E.g. the Π-matrix, the n^{th} row of which is given by the n roots of $\Pi_n(x) = \int_{-1}^{x} P_{n-1}(t)dt$ ($P_n(t)$ the Legendre-polynomial), satisfies (6) and $\theta_1^{(n)} = \theta_0^{(n)} = 0$, $\theta_n^{(n)} = \theta_{n+1}^{(n)} = \pi$. We remark that the lower bound in (10a) is implicitly contained in Fejér's paper: *Bestimmung* etc. Annali della R. Scuole Norm. Sup. Pisa, 1932.

[11] It is not uninteresting to note, that the weaker supposition $\frac{1}{n}\sum_{k=1}^{n} |l_k(x)| \leqq c_6$ is *not* sufficient to assure a quasi-uniform distribution.

We reproduce the proof of Fejér, that the matrices given by the roots of $J_n(x, \alpha, \beta)$ are strongly normal, if $0 \leqq \alpha, \beta < \frac{1}{2}$. Replacing x in (11) by a root $x_k^{(n)}$ of $J_n(x, \alpha, \beta) = 0$, we obtain

$$-\frac{\omega_n''(x_k)}{\omega_n'(x_k)} = -\frac{J_n''(x_k, \alpha, \beta)}{J_n'(x_k, \alpha, \beta)} = \frac{2(\alpha - \beta) - 2(\alpha + \beta)x_k}{1 - x_k^2}$$

further from (5c)

$$\theta_k(1) = 1 - 2(\alpha + \beta) + \frac{4\alpha}{1 + x_k} \geqq 1 - 2\beta \tag{12a}$$

and similarly

$$\theta_k(-1) = 1 - 2(\alpha + \beta) + \frac{4\beta}{1 - x_k} \geqq 1 - 2\alpha \tag{12b}$$

and (5c) immediately follows from (12a) and (12b).

Now applying theorem I we see, that the roots of $J_n(x, \alpha, \beta) = 0$ are quasi-uniformly distributed on the unit-circle in the sense of (10 b), if $0 \leqq \alpha, \beta < \frac{1}{2}$.[12] For $\alpha = \beta$ we obtain the so called ultraspherical polynomials. If $\alpha = \beta = 0$, we have the polynomial $\Pi_n(x)$ (see footnote[10]). Hence by Rolle's theorem we obtain, that the roots of the Legendre-polynomial $P_n(x)$ are quasi-uniformly distributed. Further as $J_n(x, \alpha, \beta)$ differs form $J'_{n+1}(x, \alpha - \frac{1}{2}, \beta - \frac{1}{2})$ only by a constant factor, we conclude by repeatedly employing Rolle's theorem, that the distribution of the roots of the ultraspherical polynomials is for *any* $\alpha \geqq 0$ quasi-uniform.

By this method we can obtain general results concerning the distribution of the roots of certain polynomials satisfying suitable differential equations of the second order.

In §3 we infer the structure of the matrix from properties of the Cotes-numbers. Theorem II states that *if the Cotes-numbers are non negative, then*

$$\theta_{\nu+1}^{(n)} - \theta_\nu^{(n)} \leqq \frac{c_7}{n}, \qquad \nu = 0, 1, \cdots n, \quad n = 1, 2, \cdots. \tag{13}$$

Theorem III states that *if there exists an integrable $s(x)$ lying between two positive bounds and such that*

$$\int_{-1}^{1} l_k(x) s(x)\, dx \geqq 0, \qquad k = 1, \cdots n, \quad n = 1, 2, \cdots, \tag{14}$$

[12] In addition to (11) and theorem I we must know here that each of the roots lies in $[-1, 1]$; but this is a well known elementary consequence of the orthogonality of the Jacobi polynomials with the weight function $(1 - x)^{2\alpha - 1}(1 + x)^{2\beta - 1}$.

then (13) *holds.* It will be sufficient to prove theorem III since it is more general than theorem II.[13]

Let us give an application of theorem III. Let $p(x)$ be an R-integrable function, lying between two positive bounds. Consider the orthogonal polynomials with respect to $p(x)$. As it is known, the n^{th} polynomial has in $[-1, +1]$ n different roots. Since

$$\int_{-1}^{1} l_\nu(x)p(x)\,dx = \int_{-1}^{1} l_\nu(x)^2 p(x)\,dx > 0,$$

the hypothesis of theorem III are satisfied and we obtain the

COROLLARY. *Let $p(x)$ be the weight function defined above; then denoting by $\cos \theta_\nu^{(n)}$ $(\nu = 1, 2, \cdots n)$ the roots of the n^{th} orthogonal polynomials with respect to $p(x)$, we have*

$$\theta_{\nu+1}^{(n)} - \theta_\nu^{(n)} \leqq c_8/n, \qquad \nu = 0, 1, \cdots n.$$

Combining theorem III with lemma III of §3 and replacing the η of this lemma by a $\theta_j^{(n)}(4/n < \theta_j^{(n)} < c_9/n)$ we see that, if (14) holds for a matrix, then

$$\mu_1^{(n)} \equiv \mu_1 = \int_{-1}^{1} l_1(x)s(x)\,dx < \frac{c_9}{n^2}.$$

Since

$$\sum_{k=1}^{n} \int_{-1}^{1} l_k(x)s(x)\,dx \geqq 2 \min_{|x|\leqq 1} s(x),$$

it is immediately clear, that for $n > c_{10}$ the $\mu_k^{(n)}$ cannot all be equal. For $s(x) = 1$ Bernstein[14] proved this for any $n > 9$. It would be easy to estimate $c_{10} = c_{10}(s)$ for general $s(x)$, but this we omit for the present. It is essential that $s(x)$ should be bounded; for if $p(x) = (1 - x^2)^{-\frac{1}{2}}$, then $\mu_k^{(n)} = \pi/n$, $k = 1, 2, \cdots n$.

Let us examine the effect of the interpolation property (4), or more simply that of the weaker hypotheses

$$|\lambda_k^{(n)}| \equiv \left| \int_{-1}^{1} l_k(x)\,dx \right| \leqq c_{11} n^{c_{12}}, \qquad k = 1, 2, \cdots n \tag{15}$$

---

[13] It is easy to see that from the fact that the Cotes-numbers are non-negative we cannot obtain a lower estimate for the consecutive θ's. For consider the matrix such that its $(2\nu+1)^{\text{th}}$ row is given by the roots of the $(2\nu+1)^{\text{th}}$ Legendre-polynomial, $P_{2\nu+1}(x)(\nu = 0, 1, \cdots)$ and its $2\nu^{\text{th}}$ row by the roots of $P_\nu(x)\,P_\nu[(1 + \frac{1}{2}\epsilon_\nu)x + \frac{1}{2}\epsilon_\nu]$, where $\epsilon_\nu > 0$ and so small that $(1 - \frac{1}{2}\epsilon_\nu)/(1 + \frac{1}{2}\epsilon_\nu)$ is greater than the greatest root of $P_\nu(x)$. It is easy to prove that the Cotes-numbers belonging to the matrix are all non-negative, but the difference of certain pairs of consecutive roots of the $2\nu^{\text{th}}$ row are less than ϵ_ν (and the difference of the θ's belonging to this pair $< 2\epsilon_\nu$).

[14] S. Bernstein, Comptes Rendus 1936, 1305-6.

Theorem IV asserts that

$$\theta_{\nu+1}^{(n)} - \theta_{\nu}^{(n)} \leqq \frac{c_{13} \log (n+1)}{n}\,;^{15} \qquad \nu = 0, \cdots n, \quad n = 1, 2, \cdots. \tag{16}$$

c_{13} depends *only* upon c_{11} and c_{12}. This estimate is the best possible, (16) cannot be improved even, if (4) holds. Take e.g. the matrix T, but multiply the elements of its n^{th} row by $1 - a\,\frac{\log^2 (n+1)}{n^2}$. Then the elements of the n^{th} row will be

$$\left[1 - a\,\frac{\log^2 (n+1)}{n^2}\right] \cos \frac{2l-1}{2n}\,\pi, \qquad l = 1, 2, \cdots n,$$

where $a < \frac{1}{2}$. It can be shown by simple computation that (4) holds and

$$|\,\theta_0^{(n)} - \theta_1^{(n)}\,| > c_{14}\,\frac{\log (n+1)}{n}, \qquad n = 1, 2, \cdots,$$

where c_{14} depends only upon a. We omit the details.

A Corollary of Theoem IV. *Consider a sequence of polynomials orthogonal with respect to the weight-function $p(x)$, where we only suppose that $p(x) \geqq 0$ and $\int_{-1}^{1} p(x)\,dx$ and $\int_{-1}^{1} [p(x)]^{-1}\,dx$ (in Riemann or Lebesgue-sense) exist. In this case, as we proved in I., (4) is satisfied, hence for the roots of the n^{th} polynomial* (16) *holds.*

General results about the distribution of the roots of the orthogonal-polynomials—as far as we know—are due to Szegö[16] and Bernstein.[14] They give asymptotic formulae for the orthogonal-polynomials but, as a matter of fact, they are compelled to make strong restrictions with regard to the weight-function. For these weight functions they determine asymptotically the roots, whereas our corollaries deduced from theorem III and IV give weaker, but more general results.[17]

As we saw in footnote,[13] it is impossible to give an estimate from below in theorem IV, but for essentially positive and R-integrable weights we have[18]

$$\theta_{\nu+1}^{(n)} - \theta_{\nu}^{(n)} > \frac{c_{15}}{n^2}. \qquad \nu = 1, 2, \cdots (n-1)$$

---

[15] From [13] and (4) it follows, that here too $\theta_{\nu+1}^{(n)} - \theta_{\nu}^{(n)}$ can be arbitrarily small.

[16] G. Szegö: *Über den asymptotischen Ausdruck von Polynomen* etc. Math. Ann. 1922, Bd. 86, pp. 114–139.

[17] Since, then, we have proved by another method, that if $p(x) \geqq m > 0$ is L or R-integrable, then it is possible to cover the points of infinity of $p(x)$ by intervals of total length less than ϵ so that on the remaining set the roots of the polynomials orthogonal with respect to $p(x)$ are quasi-uniformly distributed in the sense of (10b). We intend to publish in another paper this result together with others concerning the uniformly dense distribution of the fundamental points.

[18] G. Grünwald and P. Turán: *Über Interpolation.* This paper will appear in Annali di Pisa. In $[-1+\epsilon, 1-\epsilon]$ we can replace the exponent 2 by $\frac{3}{2}$.

We do not give the proof of this result here.

We have already seen that the property given by (3) does not secure the quasi-uniform distribution of the roots; but we may assert a property connected with the integral of the fundamental functions by which it is involved. The property in question is

$$\int_{-1}^{1} \frac{l_k(x)^2}{\sqrt{(1-x^2)}}\,dx \leqq \frac{c_{16}}{n}, \qquad k = 1, 2, \cdots n, \quad n = 1, 2, \cdots. \tag{17}$$

The proof of this statement is so simple, that we give it immediately. From (17) it follows, that the fundamental functions are uniformly bounded with respect to n i.e. if $1 \leqq \nu \leqq n$,

$$\max_{-1\leq x\leq +1} |\, l_\nu(x)\,| = |\, l_\nu(\xi_0)\,| = |\, l_\nu(\cos\varphi_0)\,|,$$

where without any loss of generality

$$0 \leqq \varphi_0 \leqq \frac{\pi}{2},$$

by Bernstein's theorem

$$\frac{c_{16}}{n} \geqq \int_{-1}^{1} \frac{l_\nu(x)^2}{\sqrt{(1-x^2)}}\,dx = \int_0^\pi l_\nu(\cos\theta)^2\,d\theta > \int_{\varphi_0}^{\varphi_0+\frac{1}{2n}} l_\nu(\cos\theta)^2\,d\theta > \frac{c_{17}}{n}\, l_\nu(\cos\varphi_0)^2,$$

which means, that our assertion is an immediate consequence of theorem I.

In §4, we shall be concerned with the interpolation property (8) and with the consequences of the much more general supposition

$$|\, h_k(x)\,| \leqq c_{18}\,, \quad -1 \leqq x \leqq 1, \qquad k = 1, 2, \cdots n, \quad n = 1, 2, \cdots. \tag{18}$$

In our theorem V we show, that even (18) implies quasi-uniform distribution as does (6) in theorem I. It is probable that (18) implies (6), but we cannot prove it.

2.

THEOREM I. *If*

$$|\, l_k(x)\,| \leqq c_{19}\,, \qquad -1 \leqq x \leqq +1, \qquad k = 1, \cdots n, \quad n = 1, 2, \cdots,$$

then we have

$$\frac{c_{20}}{n} \leqq \theta_{\nu+1}^{(n)} - \theta_\nu^{(n)} \leqq \frac{c_{21}}{n} \qquad \nu = 1, 2, \cdots, (n-1),$$

and the upper bound is valid for $\nu = 0$ and $\nu = n$.

First we prove the lower estimate. For any $1 \leqq \nu \leqq n$ we have by Rolle's theorem

$$\frac{1}{|\,\theta_{\nu+1}^{(n)} - \theta_\nu^{(n)}\,|} = \left|\frac{l_\nu(\cos\theta_\nu^{(n)}) - l_\nu(\cos\theta_{\nu+1}^{(n)})}{\theta_\nu^{(n)} - \theta_{\nu+1}^{(n)}}\right| = \left|\frac{dl_\nu(\cos\theta)}{d\theta}\right|_{\theta=\theta'}, \tag{19}$$

where θ' lies between $\theta_\nu^{(n)}$ and $\theta_{\nu+1}^{(n)}$. But by the hypothesis $l_\nu(\cos\theta)$ is a trigonometric polynomial of degree $(n-1)$, for which

$$|\, l_\nu(\cos\theta)\,| \leqq c_{19}\,.$$

Thus according to a well-known theorem of Bernstein-Fejér

$$\left|\frac{dl_\nu(\cos\theta)}{d\theta}\right| \leqq c_{19}(n-1).$$

Putting this into (19) we obtain

$$|\,\theta_{\nu+1}^{(n)} - \theta_\nu^{(n)}\,| \geqq \frac{1}{c_{19}(n-1)} \geqq \frac{c_{20}}{n}. \tag{20}$$

We now prove the upper estimate. Let

$$\max_{i=0,1,\cdots n}(\theta_{i+1}^{(n)} - \theta_i^{(n)}) = \theta_{\nu+1}^{(n)} - \theta_\nu^{(n)} = \frac{2D(n)c_{20}}{n}. \tag{21}$$

We must prove that $D(n) \leqq c_{22}$; we can suppose $D(n) \geqq 2$. Let $\frac{1}{2}(\theta_\nu^{(n)} + \theta_{\nu+1}^{(n)}) = \delta$ and

$$\varphi(\theta) = \frac{1}{n^2}\left(\frac{\sin n\,\dfrac{\theta+\delta}{2}}{\sin\dfrac{\theta+\delta}{2}}\right)^2 + \frac{1}{n^2}\left(\frac{\sin n\,\dfrac{\theta-\delta}{2}}{\sin\dfrac{\theta-\delta}{2}}\right)^2, \qquad 0 \leqq \theta \leqq \pi. \tag{22}$$

This expression is in consequence of a well known identity, a pure cosine polynomial in θ of degree $(n-1)$. Evidently

$$\varphi(\delta) \geqq 1. \tag{23}$$

Further

$$|\,\varphi(\theta)\,| \leqq \frac{4}{n^2(\theta+\delta)^2}\left[\frac{\dfrac{\theta+\delta}{2}}{\sin\dfrac{\theta+\delta}{2}}\right]^2 + \frac{4}{n^2(\theta-\delta)^2}\left[\frac{\dfrac{\theta-\delta}{2}}{\sin\dfrac{\theta-\delta}{2}}\right]^2. \tag{24}$$

Suppose first that $0 \leqq \delta \leqq \frac{\pi}{2}$; then $0 \leqq \dfrac{\theta+\delta}{2} \leqq \dfrac{3\pi}{4}$, $-\dfrac{\pi}{4} \leqq \dfrac{\theta-\delta}{2} \leqq \dfrac{\pi}{2}$. As $\left|\dfrac{\sin\alpha}{\alpha}\right| \geqq \dfrac{4}{3\pi}\,\dfrac{\sqrt{2}}{2}$ when $|\,\alpha\,| \leqq \dfrac{3\pi}{4}$, by (24) we have for $0 \leqq \theta \leqq \pi$

$$|\,\varphi(\theta)\,| \leqq \frac{9\pi^2}{2n^2}\left[\frac{1}{(\theta+\delta)^2} + \frac{1}{(\theta-\delta)^2}\right]. \tag{25}$$

As $\varphi(\theta)$ is a pure cosine polynomial, we have

$$\sum_{\mu=1}^{n} \varphi(\theta_\mu^{(n)}) l_\mu(\cos\theta) \equiv \varphi(\theta) \tag{26}$$

i.e. for $\theta = \delta$ in consequence of (23), (26), (25) and the hypotheses

$$1 \leqq |\varphi(\delta)| = \left|\sum_{\mu=1}^{n} \varphi(\theta_\mu^{(n)}) l_\mu(\cos\delta)\right| \leqq \sum_{\mu=1}^{n} |\varphi(\theta_\mu^{(n)})|\,|l_\mu(\cos\delta)|$$

(27)
$$\leqq \frac{9\pi^2}{2n^2} c_{19}\left[\sum_{\mu=1}^{n} \frac{1}{(\theta_\mu^{(n)} + \delta)^2} + \sum_{\mu=1}^{n} \frac{1}{(\theta_\mu^{(n)} - \delta)^2}\right] \equiv \frac{9c_{19}\pi^2}{2n^2}[S_1 + S_2].$$

But by (20)

(28a)
$$\theta_\mu^{(n)} \geqq (\mu - 1)\frac{c_{20}}{n}, \qquad \mu = 1, 2, \cdots n$$

(28b)
$$\delta \geqq \frac{D(n)c_{20}}{n}$$

thus

$$S_7 \equiv \sum_{\mu=1}^{n} \frac{1}{(\theta_\mu^{(n)} + \delta)^2} \leqq \sum_{\mu=1}^{n} \frac{1}{\left[(\mu - 1)\frac{c_{20}}{n} + \frac{D(n)c_{20}}{n}\right]^2} < \frac{n^2}{c_{20}^2} \sum_{\mu=0}^{\infty} \frac{1}{(\mu + D(n))^2},$$

and from $D(n) \geqq 2$ we have

(29a)
$$S_7 < \frac{n^2}{c_{20}^2} \int_{D(n)-1}^{\infty} \frac{dt}{t^2} = \frac{n^2}{c_{20}^2} \frac{1}{D(n) - 1}.$$

Further if $1 \leqq \nu \leqq n$,

$$S_2 = \sum_{\mu=1}^{\nu} \frac{1}{(\theta_\mu^{(n)} - \delta)^2} + \sum_{\mu=\nu+1}^{n} \frac{1}{(\theta_\mu^{(n)} - \delta)^2} \leqq \sum_{\mu=1}^{n} \frac{1}{\left[\frac{D(n)c_{20}}{n} + (\nu - \mu)\frac{c_{20}}{n}\right]^2}$$

(29b)
$$+ \sum_{\mu=\nu+1}^{n} \frac{1}{\left[\frac{D(n)c_{20}}{n} + (\mu - \nu - 1)\frac{c_{20}}{n}\right]^2} < \frac{2n^2}{c_{20}^2} \int_{D(n)-1}^{\infty} \frac{dt}{t^2} = \frac{2n^2}{c_{20}^2(D(n) - 1)}.$$

Putting (29a) and (29b) into (27) we obtain

$$1 \leqq \frac{9\pi^2 c_{19}}{2n^2} \cdot \frac{3n^2}{c_{20}^2[D(n) - 1]} = \frac{27\pi^2}{2} \frac{c_{19}}{c_{20}^2} \frac{1}{D(n) - 1} = \frac{27\pi^2}{2} c_{19}^3 \frac{1}{D(n) - 1}$$

i.e.

$$D(n) \leqq 1 + \frac{27\pi^2}{2} c_{19}^3 = c_{22}.$$

$$\theta_{\nu+1}^{(n)} - \theta_\nu^{(n)} \leqq \frac{2c_{22}c_{20}}{n} = \frac{c_{21}}{n}. \qquad \text{Q. e. d.}$$

For the cases $\left\{\begin{matrix} \nu = 0 \\ 0 \leqq \delta \leqq \pi/2 \end{matrix}\right\}$, $\left\{\begin{matrix} \nu = n \\ 0 \leqq \delta \leqq \pi/2 \end{matrix}\right\}$, $\{\pi/2 < \delta \leqq \pi\}$ the proof follows similar lines. Thus the result is established.

3.

THEOREM III. *If for the matrix there exists an L-integrable $s(x)$, which in $[-1, +1]$ lies between two positive bounds a and $b(a < b)$ and is such that*

$$\int_{-1}^{1} l_i(x)s(x)\,dx \geqq 0, \qquad i = 1, 2, \cdots n, \quad n = 1, 2, \cdots,$$

then

$$\theta_{\nu+1}^{(n)} - \theta_{\nu}^{(n)} \leqq \frac{c_{23}}{n}, \qquad \nu = 0, 1, \cdots n, \quad n = 1, 2, \cdots.$$

Let[19] μ be the greatest integer not exceeding $\frac{n+1}{2}$ i.e. $\mu > n/2, 0 \leqq \eta \leqq \pi/2$ and

$$f(\theta, n) = \frac{1}{\mu^4}\left[\left(\frac{\sin \mu \dfrac{\theta+\eta}{2}}{\sin \dfrac{\theta+\eta}{2}}\right)^4 + \left(\frac{\sin \mu \dfrac{\theta-\eta}{2}}{\sin \dfrac{\theta-\eta}{2}}\right)^4\right]; \tag{30}$$

then, for fixed η, $f(\theta, \eta)$ is a cosine polynomial the degree of which does not exceed $(n-1)$ and for which

$$|f(\theta, \eta)| \leqq \frac{c_{24}}{\mu^4} \cdot \frac{1}{(\theta-\eta)^4}, \qquad 0 \leqq \theta \leqq \pi. \tag{31}$$

LEMMA I. *If $\pi - 1/2\mu \geqq \eta \geqq 1/2\mu$, then*

$$\frac{c_{25}\eta}{\mu} \leqq J \equiv \int_0^{\pi} f(\theta, \eta)s(\cos\theta)\sin\theta\,d\theta \leqq \frac{c_{26}\eta}{\mu}.$$

Without loss of generality let $\eta \leqq \pi/2$. We have

$$J \geqq a\int_{\eta}^{\eta+\pi/\mu} f(\theta, \eta)\sin\eta\,d\theta \geqq \frac{2a\eta}{\pi\mu^4}\int_{\eta}^{\eta+\pi/\mu}\left(\frac{\frac{2}{\pi}\mu\dfrac{\theta-\eta}{2}}{\dfrac{\theta-\eta}{2}}\right)^4 d\theta = \frac{c_{26}\eta}{\mu}. \tag{32a}$$

Further

$$\left|\frac{1}{\mu^4}\int_0^{\pi}\left(\frac{\sin\mu\dfrac{\theta+\eta}{2}}{\sin\dfrac{\theta+\eta}{2}}\right)^4 \sin\theta\cdot s(\cos\theta)\,d\theta\right| < \left|\frac{2\cos\eta}{\mu^4}\int_0^{\pi}\frac{\sin^4\mu\dfrac{\theta+\eta}{2}}{\sin^3\dfrac{\theta+\eta}{2}}\cos\frac{\theta+\eta}{2}\,d\theta\right.$$

[19] Throughout this paragraph the c's are independent of D, η, θ and n, but dependent on a and b.

$$(32b)\qquad -\frac{\sin\eta}{\mu^4}\int_0^\pi\left(\frac{\sin\mu\dfrac{\theta+\eta}{2}}{\sin\dfrac{\theta+\eta}{2}}\right)^4\cos(\theta+\eta)\,d\theta\Bigg|\cdot b$$

$$< c_{27}\left[\frac{1}{\mu^4}\int_{1/\mu}^\infty\frac{dt}{t^3}+\frac{\eta}{\mu^4}\int_0^\pi\left(\frac{\sin\mu\dfrac{\theta+\eta}{2}}{\sin\dfrac{\theta+\eta}{2}}\right)^4 d\theta\right],$$

and

$$\left|\frac{1}{\mu^4}\int_0^\pi\left(\frac{\sin\mu\dfrac{\theta-\eta}{2}}{\sin\dfrac{\theta-\eta}{2}}\right)^4 s(\cos\theta)\sin\theta\,d\theta\right| < b\left|\frac{1}{\mu^4}\sin\eta\int_0^\pi\left(\frac{\sin\mu\dfrac{\theta-\eta}{2}}{\sin\dfrac{\theta-\eta}{2}}\right)^4\cos(\theta-\eta)\,d\theta\right.$$

$$(32c)\qquad \left.+\frac{2\cos\eta}{\mu^4}\int_0^\pi\frac{\sin^4\mu\dfrac{\theta-\eta}{2}}{\sin^3\dfrac{\theta-\eta}{2}}\cos\frac{\theta-\eta}{2}\,d\theta\right| < c_{28}\left[\frac{\eta}{\mu^4}\int_0^\pi\left(\frac{\sin\mu\dfrac{\theta-\eta}{2}}{\sin\dfrac{\theta-\eta}{2}}\right)^4 d\theta\right.$$

$$\left.+\frac{1}{\mu^4}\int_0^{\eta-1/\mu}\frac{d\theta}{(\eta-\theta)^3}+\frac{1}{\mu^4}\int_{\eta-1/\mu}^{\eta+1/\mu}\mu^3\,d\theta+\frac{1}{\mu^4}\int_{\eta+1/\mu}^\pi\frac{1}{(\theta-\eta)^3}\,d\theta\right].$$

Hence by (32b) and (32c)

$$(33)\qquad J \leqq c_{29}\left[\frac{\eta}{\mu^4}\int_0^\pi\left\{\left(\frac{\sin\mu\dfrac{\theta+\eta}{2}}{\sin\dfrac{\theta+\eta}{2}}\right)^4+\left(\frac{\sin\mu\dfrac{\theta-\eta}{2}}{\sin\dfrac{\theta-\eta}{2}}\right)^4\right\}d\theta+\frac{1}{\mu^2}\right].$$

For the integral on the right, the integrand being even, we have

$$\frac{1}{2}\int_0^{2\pi} f(\theta,\eta)\,d\theta.$$

But

$$(34)\qquad \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\sin\mu\dfrac{\theta+\eta}{2}}{\sin\dfrac{\theta+\eta}{2}}\right)^4 d\theta = \mu^2+2((\mu-1)^2+(\mu-2)^2+\cdots+1^2) < c_{30}\mu^3$$

by (22a) and by Parseval's theorem. The same holds for $\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\sin\mu\dfrac{\theta-\eta}{2}}{\sin\dfrac{\theta-\eta}{2}}\right)^4 d\theta$.

Hence by $\eta \geqq 1/\mu$, (33) and (34) Lemma I is proved.

LEMMA II. *In the interval*

$$0 \leqq \eta - 1/\mu \leqq \theta \leqq \eta + 1/\mu \leqq \pi$$

we have

$$f(\theta, \eta) \geqq c_{31}.$$

In this interval if $|\alpha| \leqq 1$,

$$f(\theta, \eta) \geqq \frac{1}{\mu^4}\left(\frac{\sin\frac{\alpha}{2}}{\sin\frac{\alpha}{2\mu}}\right)^4 \geqq c_{31}.$$

LEMMA III. *Consider the fundamental points* $\cos\theta_\nu^{(n)}$, *for which* $(0 \leqq)\ \eta - 1/\mu \leqq \theta_\nu^{(n)} \leqq \eta + 1/\mu (\leqq \pi)$. *If*

$$\mu_k \equiv \int_0^\pi l_k(\cos\theta) s(\cos\theta) \sin\theta\, d\theta \geqq 0, \qquad k = 1, 2, \cdots n,$$

then

$$S \equiv \sum_{\eta-1/\mu \leqq \theta_\nu \leqq \eta+1/\mu} \mu_\nu < c_{32}\eta/\mu.$$

By Lemma II and Lemma I we have

$$c_{31}S < \sum_{\eta-1/\mu \leqq \theta_\nu^{(n)} \leqq \eta+1/\mu} f(\theta_\nu^{(n)}, \eta)\mu_\nu \leqq \sum_{\nu=1}^{n} f(\theta_\nu^{(n)}, \eta)\mu_\nu$$

$$= \int_0^\pi f(\theta, \eta) s(\cos\theta) \sin\theta\, d\theta \leqq \frac{c_{26}\eta}{\mu}.$$

Hence

$$S < \frac{c_{26}}{c_{31}} \cdot \frac{\eta}{\mu} = \frac{c_{32}\eta}{\mu}.$$

Now we shall prove our theorem III. Let

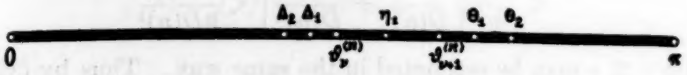
$$\max_{i=0,1,\cdots n} (\theta_{i+1}^{(n)} - \theta_i^{(n)}) = \theta_{\nu+1}^{(n)} - \theta_\nu^{(n)} = \frac{D(n)}{n+1}$$

and we have to prove that $D(n) \leqq c_{23}$. Let $\frac{1}{2}(\theta_{\nu+1}^{(n)} + \theta_\nu^{(n)}) = \eta_1$.

We may suppose without loss of generality that $\eta_1 \leqq \pi/2$ and consider the expression $f(\theta, \eta)$ of (30) for $\eta = \eta_1$. Starting at $\theta_{\nu+1}^{(n)}$ mark off to the right intervals of length $1/\mu$ till the whole of the interval $[\theta_{\nu+1}^{(n)}, \pi]$ is covered; the last interval, the length of which is less than $1/\mu$, we add to the last but one. Similarly, starting at $\theta_\nu^{(n)}$ mark off intervals to the left to cover $[0, \theta_\nu^{(n)}]$. Let the endpoints (Fig. 2) be $\Theta_1, \Theta_2, \cdots, \Delta_1, \Delta_2, \cdots$ respectively. It is evident that

$$\Theta_r = \eta_1 + \frac{D(n)}{2n+2} + \frac{r}{\mu}, \qquad r = 0, 1, 2, \cdots \tag{35a}$$

(35b) $$\Delta_s = \eta_1 - \frac{D(n)}{2n+2} - \frac{s}{\mu} \qquad s = 0, 1, 2, \cdots$$



Since

$$\sum_{i=0}^{n} (\theta_{i+1}^{(n)} - \theta_i^{(n)}) = \pi,$$

the greatest distance between two consecutive $\theta_\mu^{(n)}$'s is greater than or equal to $\pi/(n+1)$ i.e. $\eta_1 \geqq \pi/(2n+2)$, $D(n) \geqq \pi/2$. Thus by Lemma I

(36) $$\frac{c_{25}\eta_1}{\mu} \leqq \int_0^\pi f(\theta, \eta_1)s(\cos\theta)\sin\theta\, d\theta = \sum_{i=1}^{n} f(\theta_i^{(n)}, \eta_1)\mu_i.$$

In the sum on the right consider the members for which $\theta_i^{(n)}$ lies in $[\Theta_r, \Theta_{r+1}]$. For these we have by (31)

(37) $$|f(\theta_i^{(n)}, \eta_1)| \leqq \frac{c_{24}}{\mu^4}\frac{1}{(\Theta_r - \eta_1)^4}$$

and by Lemma III with $\eta = \dfrac{\Theta_r + \Theta_{r+1}}{2} = \eta_1 + \dfrac{D(n)}{2n+2} + \dfrac{r+\frac{1}{2}}{\mu}$ [20]

(38) $$\sum_{\Theta_r \leqq \theta_i^{(n)} \leqq \Theta_{r+1}} \mu_i < \frac{c_{32}\left[\eta_1 + \dfrac{D(n)}{2n+2} + \dfrac{2r+1}{2\mu}\right]}{\mu}.$$

Hence this part of the sum (36) is, by (37) and (38), less than

(39) $$\frac{c_{24}c_{32}}{\mu^5}\frac{\eta_1 + \dfrac{D(n)}{2n+2} + \dfrac{2r+1}{2\mu}}{\left(\dfrac{D(n)}{2n+2} + \dfrac{r}{\mu}\right)^4} = c_{24}c_{32}\left[\frac{\dfrac{\eta_1}{\mu^5} + \dfrac{1}{2\mu^6}}{\left(\dfrac{D(n)}{2n+2} + \dfrac{r}{\mu}\right)^4} + \frac{1}{\left(\dfrac{D(n)}{2n+2} + \dfrac{r}{\mu}\right)^3}\cdot\frac{1}{\mu^5}\right].$$

This holds evidently also for $[\theta_{r+1}^{(n)}, \Theta_1]$ with $r = 0$. Hence the partial sum of (36) summed over all $i \geqq \nu + 1$ is in consequence of $\frac{1}{2}(n+1) \geqq \mu \geqq n/2$ less than

$$c_{33}\left[\frac{\eta_1}{n}\sum_{r=0}^{\infty}\frac{1}{(D(n)+4r)^4} + \frac{1}{n^2}\sum_{r=0}^{\infty}\frac{1}{(D(n)+r)^4} + \frac{1}{n^2}\sum_{r=0}^{\infty}\frac{1}{(D(n)+r)^3}\right]$$

[20] It is clear, that for each such η, $\eta \geqq 1/2\mu$ is satisfied.

and by $\eta_1 \geqq \pi/2n$

$$< \frac{c_{34}\eta_1}{n}\left[\frac{1}{D(n)^3} + \frac{2}{D(n)^2}\right] < \frac{3c_{34}\eta_1}{nD(n)^2}. \tag{40}$$

The sum for $i \leqq \nu$ may be estimated in the same way. Thus by (36) and (40) we obtain

$$\frac{c_{25}\eta_1}{\mu} \leqq \frac{3c_{34}\eta_1}{nD(n)^2},$$

$$D(n) \leqq c_{23}. \qquad \text{Q. e. d.}$$

THEOREM IV. *If for a matrix we have*

$$\left|\int_{-1}^{1} l_\nu(x)\,dx\right| \leqq c_{11}n^{c_{12}},$$

then

$$\theta_{\nu+1}^{(n)} - \theta_\nu^{(n)} \leqq \frac{c_{13}\log(n+1)}{n}, \qquad \nu = 0, 1, \cdots n, \quad n = 1, 2, \cdots.$$

PROOF. Let $n \geqq 9$, r be even and $\geqq 4$, the odd integer $m \geqq 5$ so that

$$\tfrac{1}{2}(m-1)r \leqq n-1; \tag{41}$$

r and m are indefinite for the moment with only the restriction that both tend to infinity as $n \to \infty$. Let

$$\max_{i=0,1,\cdots n} (\theta_{i+1}^{(n)} - \theta_i^{(n)}) = \theta_{\nu+1}^{(n)} - \theta_\nu^{(n)} = 2\delta(n)$$

$$\tfrac{1}{2}(\theta_{\nu+1}^{(n)} + \theta_\nu^{(n)}) = \theta',$$

and

$$f(\theta) = \frac{1}{m^r}\left[\left(\frac{\sin m\dfrac{\theta+\theta'}{2}}{\sin\dfrac{\theta+\theta'}{2}}\right)^r + \left(\frac{\sin m\dfrac{\theta-\theta'}{2}}{\sin\dfrac{\theta-\theta'}{2}}\right)^r\right]. \tag{42}$$

Evidently $f(\theta)$ is a trigonometric polynomial of degree $\tfrac{1}{2}(m-1)r \leqq n-1$ and as $f(\theta) = f(-\theta)$, it is a pure cosine polynomial. First consider $0 \leqq \theta' \leqq \pi/2$; if $\pi/2 < \theta' \leqq \pi$, the proof is similar. As $\delta(n) \geqq \pi/(2n+2)$ we have $\theta' \geqq \pi/(2n+2)$.

As $m \geqq 5$, the interval $[\theta', \theta' + \pi/m]$ will lie entirely in $[0, \pi]$.

Then, as r is even and $\theta' \leqq \pi/2$,

$$\int_0^\pi f(\theta)\sin\theta\,d\theta \geqq \frac{1}{m^r}\int_{\theta'}^{\theta'+\pi/m}\left(\frac{\dfrac{2}{\pi}m\dfrac{\theta-\theta'}{2}}{\dfrac{\theta-\theta'}{2}}\right)^r \frac{2}{\pi}\delta(n)\,d\theta = \left(\frac{2}{\pi}\right)^r\frac{2}{m}\delta(n). \tag{43}$$

The degree of $f(\theta)$ being $(n-1)$ at most, we have

$$\sum_{i=1}^{n} f(\theta_i^{(n)}) l_i(\cos\theta) \equiv f(\theta),$$

hence by (43)

$$2\left(\frac{2}{\pi}\right)^r \frac{\delta(n)}{m} \leqq \int_0^\pi f(\theta)\sin\theta\, d\theta = \left|\sum_{i=1}^{n} f(\theta_i^{(n)}) \int_0^\pi l_i(\cos\theta)\sin\theta\, d\theta\right|$$

(44)

$$= \left|\sum_{i=1}^{n} f(\theta_i^{(n)}) \int_{-1}^{1} l_i(x)\, dx\right| \leqq c_{11} n^{c_{12}} \cdot n \max_{i=1,\cdots n} |f(\theta_i^{(n)})|.$$

As $|\frac{1}{2}(\theta - \theta')| \leqq \pi/2$, we have from (42)

$$|f(\theta_i^{(n)})| \leqq \frac{1}{m^r}\left[\frac{1}{\left[\frac{1}{\pi}(\theta_i^{(n)} - \theta')\right]^r} + \frac{1}{\left[\frac{1}{\pi}\delta(n)\right]^r}\right] \leqq 2\left(\frac{\pi}{m\delta(n)}\right)^r; \tag{45}$$

by putting (45) into (44)

$$2\left(\frac{2}{\pi}\right)^r \frac{\delta(n)}{m} \leqq c_{11} n^{1+c_{12}} 2\left(\frac{\pi}{m\delta(n)}\right)^r,$$

thus

$$\delta(n) \leqq \frac{\pi^2}{m}\left(\frac{2c_{11}}{\pi^2} m^2 n^{1+c_{12}}\right)^{\frac{1}{r+1}}. \tag{46}$$

Now let r be the greatest even integer not exceeding $\log n$ and m be the greatest integer less than $(2n-2)/r$; then (41) is satisfied for sufficiently great n and thus by (46) we have

$$\delta(n) \leqq \frac{c_{13}\log(n+1)}{n},$$

which establishes the result.

4.

Here we have to prove that from

$$|\theta_k(x)|\, l_k(x)^2 \equiv |h_k(x)| \leqq c_{36} \tag{47}$$

$$-1 \leqq x \leqq +1, \qquad k = 1, 2, \cdots n, \qquad n = 1, 2, \cdots,$$

follows

$$\frac{c_{37}}{n} \leqq \theta_{\nu+1}^{(n)} - \theta_\nu^{(n)} \leqq \frac{c_{38}}{n}.$$ [21]

[21] The upper estimate holds for $\nu = 0, 1, 2 \cdots n$, the lower for $\nu = 1, 2, \cdots (n-1)$.

We obtain the lower estimate as in theorem I, namely

$$\frac{1}{|\theta_{\nu+1}^{(n)}-\theta_\nu^{(n)}|}=\left|\frac{h_\nu(\cos\theta_\nu^{(n)})-h_\nu(\cos\theta_{\nu+1}^{(n)})}{\theta_{\nu+1}^{(n)}-\theta_\nu^{(n)}}\right|=\left|\frac{dh_\nu(\cos\theta)}{d\theta}\right|_{\theta=\varphi_1}$$

where according to Rolle's theorem φ_1 lies between $\theta_\nu^{(n)}$ and $\theta_{\nu+1}^{(n)}$. Since $h_\nu(\cos\theta)$ is a trigonometric polynomial of degree $(2n-1)$, we obtain by Bernstein's theorem

$$\frac{1}{|\theta_{\nu+1}^{(n)}-\theta_\nu^{(n)}|}\leqq(2n-1)\max_{-1\leqq x\leqq 1}|h_\nu(x)|\leqq c_{36}(2n-1),$$

hence

$$\theta_{\nu+1}^{(n)}-\theta_\nu^{(n)}\geqq\frac{1}{c_{36}(2n-1)}>\frac{c_{37}}{n},\qquad \nu=1,2,\cdots(n-1). \tag{48}$$

Let us now consider the upper bound. Let

$$\max_{i=0,1,\cdots n}(\theta_{i+1}^{(n)}-\theta_i^{(n)})=\theta_{\mu+1}^{(n)}-\theta_\mu^{(n)}=\frac{2D(n)}{n+1}, \tag{49a}$$

and

$$\tfrac{1}{2}(\theta_\mu^{(n)}+\theta_{\mu+1}^{(n)})=\varphi_2. \tag{49b}$$

From

$$\sum_{\nu=0}^{n}(\theta_{\nu+1}^{(n)}-\theta_\nu^{(n)})=\pi$$

we obtain

$$D(n)\geqq\pi/2. \tag{49c}$$

Without any loss of generality we may suppose

$$0\leqq\varphi_2\leqq\pi/2. \tag{49d}$$

Let $\varphi(x)$ be the polynomial (its degree does not exceed $(n-1)$), for which $\varphi(\cos\theta)$ is identical with the polynomial defined at (42) if we replace θ' by φ_2 and $r=10$, $m=\left[\frac{n-1}{5}\right]$. Since

$$|\phi(\cos\theta_\nu^{(n)})|<\frac{c_{39}}{n^{10}|\theta_\nu^{(n)}-\varphi_2|^{10}}, \tag{50}$$

we have

$$1\leqq\phi(\cos\varphi_2)=\sum_{\nu=1}^{n}\phi(\cos\theta_\nu^{(n)})\,l_\nu(\cos\varphi_2)<\frac{c_{39}}{n^{10}}\sum_{\nu=1}^{n}\frac{|l_\nu(\cos\varphi_2)|}{|\theta_\nu^{(n)}-\varphi_2|^{10}}. \tag{51}$$

CASE I. $|l_\nu(\cos\varphi_2)|\leqq n^8|\theta_\nu^{(n)}-\varphi_2|^8$, $\nu=1,2,\cdots n$.

From (51) we have

$$1 \leqq \frac{c_{39}}{n^2} \sum_{\nu=1}^{n} \frac{1}{|\theta_\nu^{(n)} - \varphi_2|^2} \tag{52a}$$

and by (48)

$$|\theta_\nu^{(n)} - \varphi_2| \geqq \frac{D(n) + |\mu - \nu| c_{37}}{n+1} c_{40}. \tag{52b}$$

Finally by putting (52b) into (52a) we obtain

$$1 \leqq c_{41} \sum_{\nu=\mu}^{\infty} \frac{1}{[D(n) + c_{37}(\nu - \mu)]^2} < \frac{c_{42}}{D(n)},$$

$$D(n) \leqq c_{38},$$

which settles case I.

CASE II. A. There is a k such, that

$$|l_k(\cos \varphi_2)| > n^8 (\theta_k^{(n)} - \varphi_2)^8 \tag{53}$$

and $l_k(\cos \theta)$ takes its absolute maximum in $[\theta_\mu^{(n)}, \theta_{\mu+1}^{(n)}]$. First we require two lemmas.

LEMMA 1. *Let $f(\theta)$ be a cosine polynomial of degree m, the roots of which are all real and distinct*

$$(\psi_0 \equiv) 0 \leqq \psi_1 < \psi_2 < \cdots < \psi_m \leqq \pi (\equiv \psi_{m+1}),$$

taking its absolute maximum in $[\psi_\nu, \psi_{\nu+1}]$; then to every ξ in $[\psi_\nu, \psi_{\nu+1}]$ there exists an interval l such that:

1. *l lies in $[\psi_\nu, \psi_{\nu+1}]$,*
2. *ξ is an endpoint of l,*
3. *The length of l is greater than $1/2m$,*
4. *For every θ lying in l we have*

$$|f(\theta)| > \tfrac{1}{2} |f(\xi)|.$$

PROOF. According to the hypothesis $f(\theta)$ has in $[\psi_\nu, \psi_{\nu+1}]$ the unique extreme $\theta = \varphi_3$; we can suppose this to be a maximum. Suppose first $\xi \leqq \varphi_3$. If $\xi + 1/2m \leqq \varphi_3$; our lemma follows from the fact, that $f(\theta)$ is monotonously increasing in $[\xi, \varphi_3]$.

Suppose now $\xi + 1/2m > \varphi_3$. Then Bernstein's well-known theorem gives

$$|f'(\theta)| \leqq m f(\varphi_3)$$

and from this we have

$$f\left(\xi + \frac{1}{2m}\right) = f(\varphi_3) + \int_{\varphi_3}^{\xi+1/2m} f'(\theta)\, d\theta \tag{54}$$

$$> f(\varphi_3) - \frac{1}{2m} \cdot m f(\varphi_3) = \tfrac{1}{2} f(\varphi_3) \geqq \tfrac{1}{2} f(\xi).$$

We obtain the lower estimate as in theorem I, namely

$$\frac{1}{|\theta_{\nu+1}^{(n)} - \theta_\nu^{(n)}|} = \left| \frac{h_\nu(\cos\theta_\nu^{(n)}) - h_\nu(\cos\theta_{\nu+1}^{(n)})}{\theta_{\nu+1}^{(n)} - \theta_\nu^{(n)}} \right| = \left| \frac{dh_\nu(\cos\theta)}{d\theta} \right|_{\theta=\varphi_1}$$

where according to Rolle's theorem φ_1 lies between $\theta_\nu^{(n)}$ and $\theta_{\nu+1}^{(n)}$. Since $h_\nu(\cos\theta)$ is a trigonometric polynomial of degree $(2n-1)$, we obtain by Bernstein's theorem

$$\frac{1}{|\theta_{\nu+1}^{(n)} - \theta_\nu^{(n)}|} \leqq (2n-1) \max_{-1\leqq x\leqq 1} |h_\nu(x)| \leqq c_{36}(2n-1),$$

hence

$$\theta_{\nu+1}^{(n)} - \theta_\nu^{(n)} \geqq \frac{1}{c_{36}(2n-1)} > \frac{c_{37}}{n}, \qquad \nu = 1, 2, \cdots (n-1). \tag{48}$$

Let us now consider the upper bound. Let

$$\max_{i=0,1,\cdots n} (\theta_{i+1}^{(n)} - \theta_i^{(n)}) = \theta_{\mu+1}^{(n)} - \theta_\mu^{(n)} = \frac{2D(n)}{n+1}, \tag{49a}$$

and

$$\tfrac{1}{2}(\theta_\mu^{(n)} + \theta_{\mu+1}^{(n)}) = \varphi_2. \tag{49b}$$

From

$$\sum_{\nu=0}^{n} (\theta_{\nu+1}^{(n)} - \theta_\nu^{(n)}) = \pi$$

we obtain

$$D(n) \geqq \pi/2. \tag{49c}$$

Without any loss of generality we may suppose

$$0 \leqq \varphi_2 \leqq \pi/2. \tag{49d}$$

Let $\varphi(x)$ be the polynomial (its degree does not exceed $(n-1)$), for which $\varphi(\cos\theta)$ is identical with the polynomial defined at (42) if we replace θ' by φ_2 and $r = 10, m = \left[\frac{n-1}{5}\right]$. Since

$$|\phi(\cos\theta_\nu^{(n)})| < \frac{c_{39}}{n^{10}|\theta_\nu^{(n)} - \varphi_2|^{10}}, \tag{50}$$

we have

$$1 \leqq \phi(\cos\varphi_2) = \sum_{\nu=1}^{n} \phi(\cos\theta_\nu^{(n)}) l_\nu(\cos\varphi_2) < \frac{c_{39}}{n^{10}} \sum_{\nu=1}^{n} \frac{|l_\nu(\cos\varphi_2)|}{|\theta_\nu^{(n)} - \varphi_2|^{10}}. \tag{51}$$

CASE I. $|l_\nu(\cos\varphi_2)| \leqq n^8 |\theta_\nu^{(n)} - \varphi_2|^8$, $\nu = 1, 2, \cdots n$.

From (51) we have

$$1 \leqq \frac{c_{39}}{n^2} \sum_{\nu=1}^{n} \frac{1}{|\theta_\nu^{(n)} - \varphi_2|^2} \tag{52a}$$

and by (48)

$$|\theta_\nu^{(n)} - \varphi_2| \geqq \frac{D(n) + |\mu - \nu| c_{37}}{n+1} c_{40}. \tag{52b}$$

Finally by putting (52b) into (52a) we obtain

$$1 \leqq c_{41} \sum_{\nu=\mu}^{\infty} \frac{1}{[D(n) + c_{37}(\nu - \mu)]^2} < \frac{c_{42}}{D(n)},$$

$$D(n) \leqq c_{38},$$

which settles case I.

CASE II. A. There is a k such, that

$$|l_k(\cos \varphi_2)| > n^8 (\theta_k^{(n)} - \varphi_2)^8 \tag{53}$$

and $l_k(\cos \theta)$ takes its absolute maximum in $[\theta_\mu^{(n)}, \theta_{\mu+1}^{(n)}]$. First we require two lemmas.

LEMMA 1. *Let $f(\theta)$ be a cosine polynomial of degree m, the roots of which are all real and distinct*

$$(\psi_0 \equiv) 0 \leqq \psi_1 < \psi_2 < \cdots < \psi_m \leqq \pi (\equiv \psi_{m+1}),$$

taking its absolute maximum in $[\psi_\nu, \psi_{\nu+1}]$; then to every ξ in $[\psi_\nu, \psi_{\nu+1}]$ there exists an interval l such that:

1. *l lies in $[\psi_\nu, \psi_{\nu+1}]$,*
2. *ξ is an endpoint of l,*
3. *The length of l is greater than $1/2m$,*
4. *For every θ lying in l we have*

$$|f(\theta)| > \tfrac{1}{2} |f(\xi)|.$$

PROOF. According to the hypothesis $f(\theta)$ has in $[\psi_\nu, \psi_{\nu+1}]$ the unique extreme $\theta = \varphi_3$; we can suppose this to be a maximum. Suppose first $\xi \leqq \varphi_3$. If $\xi + 1/2m \leqq \varphi_3$; our lemma follows from the fact, that $f(\theta)$ is monotonously increasing in $[\xi, \varphi_3]$.

Suppose now $\xi + 1/2m > \varphi_3$. Then Bernstein's well-known theorem gives

$$|f'(\theta)| \leqq m f(\varphi_3)$$

and from this we have

$$f\left(\xi + \frac{1}{2m}\right) = f(\varphi_3) + \int_{\varphi_3}^{\xi+1/2m} f'(\theta)\, d\theta \tag{54}$$

$$> f(\varphi_3) - \frac{1}{2m} \cdot m f(\varphi_3) = \tfrac{1}{2} f(\varphi_3) \geqq \tfrac{1}{2} f(\xi).$$

As (54) a fortiori holds for $\xi \leqq \theta \leqq \xi + 1/2m$, the lemma is proved for $\xi \leqq \varphi_3$. Similarly for $\xi > \varphi_3$ we consider the interval $[\xi - 1/2m, \xi]$.

Let us now consider the case II A. Since $l_k(\cos\theta)$ takes its absolute maximum in $[\theta_\mu^{(n)}, \theta_{\mu+1}^{(n)}]$, we obtain from our lemma by putting $f(\theta) = l_k(\cos\theta)$, $\xi = \varphi_2$, and (53), that in $[\varphi_2, \varphi_2 \pm 1/(2n-2)]$[22] and a fortiori in $[\varphi_2, \varphi_2 \pm 1/2n]$

$$|l_k(\cos\theta)| \geqq \tfrac{1}{2}|l_k(\cos\varphi_2)| > \frac{n^8}{2}|\theta_k^{(n)} - \varphi_2|^8.$$

Thus for $\varphi_2 \pm 1/2n \equiv \varphi_4$[23] we have

$$|l_k(\cos\varphi_4)| > \frac{n^8}{2}|\theta_k^{(n)} - \varphi_2|^8. \tag{55}$$

A simple geometrical observation shows that if for the linear function $a(x) = \alpha x + \beta$

$$a(\xi_1) = 1,$$

and, further, ξ_2 and ξ_3 lie on the same side of ξ_1, then

$$\max(|a(\xi_2)|, |a(\xi_3)|) \geqq \frac{\left|\frac{\xi_2 - \xi_3}{2}\right|}{\left|\frac{\xi_2 + \xi_3}{2} - \xi_1\right|} = \frac{\left|\frac{\xi_2 + \xi_3}{2} - \xi_3\right|}{\left|\frac{\xi_2 + \xi_3}{2} - \xi_1\right|}. \tag{56}$$

By applying (56) to $a(x) \equiv \theta_k(x)$ and putting $\xi_1 = \cos\theta_k^{(n)}$, $\xi_2 = \cos\varphi_2$, $\xi_3 = \cos\varphi_4$ we obtain

$$\max(|\theta_k(\cos\varphi_2)|, |\theta_k(\cos\varphi_4)|) \geqq \frac{\left|\frac{\cos\varphi_2 + \cos\varphi_4}{2} - \cos\varphi_4\right|}{\left|\frac{\cos\varphi_2 + \cos\varphi_4}{2} - \cos\theta_k^{(n)}\right|}. \tag{57}$$

Replacing in (57) $\frac{1}{2}(\cos\varphi_2 + \cos\varphi_4)$ by $\cos\varphi_5$ ($\varphi_2 \leqq \varphi_5 \leqq \varphi_4$ or $\varphi_4 \leqq \varphi_5 \leqq \varphi_2$) we have

$$\max(|\theta_k(\cos\varphi_2)|, |\theta_k(\cos\varphi_4)|) \geqq \left|\frac{\cos\varphi_5 - \cos\varphi_4}{\cos\varphi_5 - \cos\theta_k^{(n)}}\right|.^{24} \tag{58}$$

Now we prove

LEMMA 2. *Let* $0 \leqq \lambda_1 < \lambda_2 < \lambda_3 \leqq \pi$, *then*

$$\left|\frac{\cos\lambda_1 - \cos\lambda_2}{\cos\lambda_1 - \cos\lambda_3}\right| \geqq \frac{4}{\pi^2}\left(\frac{\lambda_1 - \lambda_2}{\lambda_1 - \lambda_3}\right)^2.^{25} \tag{59}$$

---

[22] The sign which we must take depends only on the position of φ_2.

[23] From the lemma it follows that φ_4 also lies in $[\theta_\mu^{(n)}, \theta_{\mu+1}^{(n)}]$.

[24] It is clear, that the numerator can also be written in the form $|\frac{1}{2}(\cos\varphi_2 + \cos\varphi_4) - \cos\varphi_3|$. We use this form if φ_3 lies between φ_5 and $\theta_k^{(n)}$; if φ_4 lies between φ_5 and $\theta_k^{(n)}$ we use the form (58).

[25] Obviously one and only one of these two cases arises; we suppose in the text the second one. Evidently the same inequality holds for $0 \leqq \lambda_3 < \lambda_2 < \lambda_1 \leqq \pi$.

PROOF. Obviously if $\angle\, 0_1 0 A_\nu = \lambda_\nu$, $|\cos\lambda_\nu - \cos\lambda_\mu|$ is the projection of $A_\mu A_\nu$; then for fixed $\lambda_2 - \lambda_1$ and $\lambda_3 - \lambda_1$ the quotient $\left|\frac{\cos\lambda_1 - \cos\lambda_2}{\cos\lambda_1 - \cos\lambda_3}\right|$ takes its minimum at $\lambda_1 = 0$; the value of the minimum is evidently

$$\frac{1-\cos(\lambda_2-\lambda_1)}{1-\cos(\lambda_3-\lambda_1)} = \left(\frac{\sin\frac{\lambda_2-\lambda_1}{2}}{\sin\frac{\lambda_3-\lambda_1}{2}}\right)^2 \geqq \frac{4}{\pi^2}\left(\frac{\lambda_2-\lambda_1}{\lambda_3-\lambda_1}\right)^2,$$

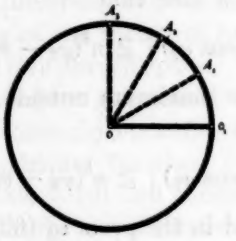


FIG. 3

which proves the lemma.

Applying lemma 2 to (58) we obtain

$$\max\left(|\theta_k(\cos\varphi_2)|, |\theta_k(\cos\varphi_4)|\right) \geqq \frac{4}{\pi^2}\left(\frac{\varphi_5-\varphi_4}{\varphi_5-\theta_k^{(n)}}\right)^2. \tag{60}$$

Now from the definition of φ_5 and by $|\varphi_2 - \varphi_4| = 1/2n$ we obtain

$$|\varphi_5 - \varphi_4| > \frac{1}{10n} \tag{61a}$$

and

$$|\varphi_5 - \theta_k^{(n)}| < |\varphi_2 - \theta_k^{(n)}|.^{26} \tag{61b}$$

Putting (61a) and (61b) into (60) we obtain

$$\max\left(|\theta_k(\cos\varphi_2)|, |\theta_k(\cos\varphi_4)|\right) \geqq \frac{1}{25\pi^2 n^2(\varphi_2-\theta_k^{(n)})^2}, \tag{62}$$

which with (53) and (55) gives

$$c_{36} \geqq \max\left(|l_k(\cos\varphi_2)|, |l_k(\cos\varphi_4)|\right) \geqq c_{43} n^{14}(\varphi_2-\theta_k^{(n)})^{14}. \tag{63}$$

---

[26] If φ_2 lies between φ_5 and $\theta_k^{(n)}$ (see footnote 24) then $|\varphi_5 - \varphi_2| > 1/10\,n$ holds instead of (61a) and $|\varphi_5 - \theta_k^{(n)}| \leqq (1+\pi^{-1})\,|\varphi_2 - \theta_k^{(n)}|$ instead of (61b), since according to (49c) and $|\varphi_3 - \varphi_2| < |\varphi_4 - \varphi_2| = 1/2\,n$,

$$\left|\frac{\varphi_5-\theta_k^{(n)}}{\varphi_2-\theta_k^{(n)}}\right| = \left|\frac{\varphi_5-\varphi_2}{\varphi_2-\theta_k^{(n)}}\right| + 1 < 1 + \frac{1}{\pi}.$$

Hence

$$|\varphi_2 - \theta_k^{(n)}| < \frac{c_{44}}{n}, \tag{64}$$

which evidently means that

$$|\theta_{\mu+1}^{(n)} - \theta_\mu^{(n)}| \leqq 2|\varphi_2 - \theta_k^{(n)}| \leqq \frac{c_{38}}{n} \tag{65}$$

and this settles II A.

CASE II B. There exists a k such that

$$|l_k(\cos \varphi_2)| \geqq n^8(\varphi_2 - \theta_k^{(n)})^8 \tag{66}$$

and $l_k(\cos\theta)$ takes its absolute maximum outside $[\theta_\mu^{(n)}, \theta_{\mu+1}^{(n)}]$, at $\theta = \varphi_6$. From the definition of φ_6 we have

$$|l_k(\cos \varphi_6)| \geqq n^8(\varphi_2 - \theta_k^{(n)})^8. \tag{67}$$

The only property of φ_4 used in the proof of (62) was that its distance from φ_2 lies e.g. between $\pi/8n$ and $\pi/4n$. Thus (62) holds here too, if φ_2 has the same meaning as in case I and instead of φ_4 we take an arbitrary point φ of the interval $[\varphi_2 \pm \pi/8n, \varphi_2 \pm \pi/4n]$.[23] We remark, that φ_6 is farther from φ_2 than φ, and note, that by using (66) and (67) the whole idea of the proof of case II A may be applied here too, if we have proved following lemma.

LEMMA 3. *Let $\psi(x)$ be linear and denote the minimum of max $(|\psi(\cos\varphi_2)|, |\psi(\cos\alpha)|)$ by $M = M(\varphi_2, \alpha, \theta_k^{(n)})$, if $\psi(x)$ runs over the lines, for which $\psi(\cos\theta_k^{(n)}) = 1$. Then M does not decrease, if $\theta_k^{(n)}$ and φ_2 are fixed and $|\alpha - \varphi_2|$ increases $(\alpha \neq \theta_k^{(n)})$.*

If φ_2 and α lie on the same side of $\theta_k^{(n)}$ and α is fixed then the minimum is attained for the straight line connecting the point $(\cos\theta_k^{(n)}, 1)$ with the bisecting point of the distance $(\cos\alpha, 0)$ and $(\cos\varphi_2, 0)$. This evidently proves the lemma for this case and it is also clear, that the minimum is less than 1. If α and φ_2 are situated on opposite sides of $\theta_k^{(n)}$, the minimum is attained if $\psi(x) \equiv 1$ and then its value is 1.

MANCHESTER, BUDAPEST.

ANNALS OF MATHEMATICS
Vol. 39, No. 4, October, 1938

HYPERSURFACES IN A PROJECTIVE CURVED SPACE

BY V. HLAVATÝ

(Received January 28, 1938)

INTRODUCTION

This paper contains, with subsequent developments, the material presented in my lectures at Princeton University during the spring term of 1938. The reader finds here some account of the theory of hypersurfaces (both holonomic and non-holonomic) in a projective curved space. I have divided the paper into three chapters. In the first chapter (sections 1–6) the necessary notions and definitions are introduced and some objects defined which are needed in the next two chapters. In the second chapter (sections 7–13) I confine myself first to the non-holonomic case (sections 7–10) and consider the correlation associated with the given non-holonomic subspace. This correlation induces a projectivity which preserves, point by point, a projective "normal" direction (of order 2) of the hypersurface. The remaining sections of this chapter deal with the family of polarities, associated with the subspace (holonomic or non-holonomic) which in the holonomic case are at least of order 3 and enable us to define a "canonical plane" (section 12). A privileged polarity of this family is considered in the last section.

In the third chapter (sections 14–18) the problem of normal direction and the connection induced by it are discussed. A normal direction (of order three in the holonomic general case) is found which, in the non-holonomic case, reduces to the normal direction mentioned above. (If the large space is flat and the hypersurface is holonomic then this normal direction is of course of order 4.) This normal direction induces a family of connections in the given hypersurface and enables us to construct an "absolute" connection. In the last section a projective Weyl's connection is introduced and discussed.

Some of the problems considered here have already been treated by Schouten and Haantjes (Compositio Math. 3, pp. 1–51, 1936) in a completely different arrangement. These authors use the homogeneous coördinates in the curved space and consequently do not start with the usual definition of a projective transformation of an affine connection which gives rise to the notion of a projective curved space as used in dealing with the Princeton algorithm.

CHAPTER I

1. Definition of $X_{n-1}^{(n)}$

Let x^{ν} [1] be the coördinates of an n-dimensional space. Consider a subspace (hypersurface) in this space, given by the parametric equations

$$x^{\nu} = x^{\nu}(y^{\mathrm{i}}, \cdots, y^{\dot{n}-\mathrm{i}}). \tag{1.1}$$

[1] It is understood that small (capital) Greek indices take the values $1, \cdots, n$ $(0, 1, \cdots, n)$.

Throughout this paper we shall consider only those points of (1, 1), where the matrix

$$\left\| \frac{\partial x^\nu}{\partial y^\alpha} \right\|$$ [2]

has the rank $n - 1$. It follows from this that the $n - 1$ contravariant vectors

$$\underset{a}{u^\nu}(y) \equiv \frac{\partial x^\nu}{\partial y^\alpha} \tag{1.2a}$$

are linearly independent. The functions $\dfrac{\partial x^\nu}{\partial y^\alpha}$ may also be thought of as the components of the mixed unity tensor, the upper index referring to the space and the lower index to the subspace. It follows from this that we may also write

$$u^\nu_a(y) \equiv \frac{\partial x^\nu}{\partial y^\alpha}, \tag{1.2b}$$

where now the position of the indices emphasizes the contra- and covariant character of the tensor.

Any non-holonomic subspace (non-holonomic hypersurface) may be given by means of $n - 1$ linearly independent vector fields $\underset{a}{w^\nu}(x)$. If $A^a_{a'} \equiv A^a_{a'}(x)$ is a set of scalars with determinant different from zero, then the same non-holonomic hypersurface is also given by $n - 1$ linearly independent vector fields

$$\underset{a'}{w^\nu}(x) = A^b_{a'} \underset{b}{w^\nu}. \tag{1.3}$$

It follows from this that $\underset{a}{w^\nu}$ may be thought of as the components of the mixed unity tensor, the upper index again referring to the space and the lower index to the non-holonomic subspace. In such a case we write $w^\nu_a(x)$ instead of $\underset{a}{w^\nu}(x)$.

The subspace (1.1) will be designated by X_{n-1}, the non-holonomic subspace by X^n_{n-1}. In order to be able to deal simultaneously with X_{n-1} and X^n_{n-1} we shall make the following conventions:

a) $X^{(n)}_{n-1}$ will designate either X_{n-1} or X^n_{n-1}.

b) $$\underset{a}{u^\nu} \equiv \begin{cases} \underset{a}{u^\nu}(y) \\ \underset{a}{w^\nu}(x) \end{cases} \quad \text{for} \quad \begin{cases} X_{n-1} \\ X^n_{n-1} \end{cases},$$

c) $$u^\nu_a \equiv \begin{cases} u^\nu_a(y) \\ w^\nu_a(x) \end{cases} \quad \text{for} \quad \begin{cases} X_{n-1} \\ X^n_{n-1} \end{cases}.$$ [3]

---

[2] Small (capital) Roman indices, with the exception of (i) and (e), run from 1 to $n - 1$ (0 to $n - 1$).

[3] By writing $\underset{a}{u^\nu}$ we emphasize that we regard this set of functions as vectors, which is sometimes very convenient in tensor analysis.

d) $$A_b^{a'} \equiv \frac{\partial y^{a'}}{\partial y^b}, \qquad A_{b'}^{a} \equiv \frac{\partial y^{a}}{\partial y^{b'}} \quad \text{for} \quad X_{n-1},$$

while in the non-holonomic case, the same symbols appear in (1.3) and have nothing to do with any transformation of parameters $y^{a'} \to y^b$. But in any case we suppose that

(1.4a) $$\Delta_{(i)} \equiv \det. \mid A_b^{a'} \mid \neq 0.$$ [4]

e) $$\partial_a \equiv \begin{cases} \dfrac{\partial}{\partial y^a} \\ u_a^\lambda \dfrac{\partial}{\partial x^\lambda} \end{cases} \quad \text{for} \quad \begin{cases} X_{n-1} \\ X_{n-1}^n \end{cases}.$$

f) By a "transformation of reference system" we understand either a transformation $y^{a'} \to y^b$ of parameters if dealing with X_{n-1}, or the transformation (1.3) in the non-holonomic case.

Throughout this paper we shall consider only such transformations $x^{\nu'} \to x^\nu$ of the x's for which

(1.4b) $$\Delta_{(e)} \equiv \det \left| \frac{\partial x^{\nu'}}{\partial x^\lambda} \right| \neq 0.$$ [4]

2. Problem to be solved

Let $\Gamma_{\lambda\mu}^\nu = \Gamma_{\mu\lambda}^\nu$ be the connection coefficients of the large space. The most general transformation of this connection, which preserves its symmetry and its paths, is the so-called "projective transformation" defined by

(2.1) $$\hat{\Gamma}_{\lambda\mu}^\nu = \Gamma_{\lambda\mu}^\nu + \delta_\lambda^\nu r_\mu + \delta_\mu^\nu r_\lambda,$$

where δ_λ^ν stands for the Kronecker delta and r_λ is any covariant vector. Any property invariant with regard to (2.1) will be referred to as a *projective property*.

The tangent plane of $X_{n-1}^{(n)}$ is a covariant vector t_λ which satisfies the set of equations

(2.2) $$u_a^\lambda t_\lambda = 0.$$

If t_λ is any solution of (2.2), the most general solution of this system is

(2.3) $$\bar{t}_\lambda = t_\lambda f,$$

where

$$f \equiv \begin{cases} f(y) \\ \varphi(x) \end{cases} \neq 0 \quad \text{for} \quad \begin{cases} X_{n-1} \\ X_{n-1}^n \end{cases}$$

[4] The index (e) [= external] in (1.4b) points out that the determinant considered here is connected with the large space, the index (i) [= internal] used in (1.4a) refers to the subspace.

represents any function of the designated arguments. Any object ϕ which transforms by (2.3) according to

$$\bar{\phi} = \epsilon \phi f^k, \qquad (\epsilon = \pm 1) \tag{2.4}$$

will be referred to as "intrinsic object of degree k"[5]

The problem to be solved may be stated in the following manner.

1) Starting with the given $X_{n-1}^{(n)}$, one has to associate with it a projective and intrinsic correlation which presents an analogy to the Darboux quadrics as known in the classical geometry. (It is well known that there are three types of correlations: a) general correlation, b) polarity, and (for n odd) c) linear complex. In this paper we are dealing only with the first two kinds of correlations, and we reserve the study of a linear complex, associated with the given X_{n-1}^{n} for another occasion.)

2) Starting with the given $X_{n-1}^{(n)}$, one has to define a projective and intrinsic normal direction to it and to consider the connection induced by this normal direction in $X_{n-1}^{(n)}$.

3. Projective objects in $\mathfrak{P}_n$

Following the idea of T. Y. Thomas[6] we introduce the set of functions connected with the transformations $x^{\nu'} \to x^{\nu}$ and defined in the following way

$$\begin{aligned} &P_0^{0'} \equiv 1, \quad P_0^{\nu'} \equiv 0, \quad P_\lambda^{\nu'} \equiv A_\lambda^{\nu'} \equiv \frac{\partial x^{\nu'}}{\partial x^\lambda}, \quad P_\lambda^{0'} \equiv -\frac{1}{c(n+1)} \frac{\partial \log \Delta_{(\epsilon)}}{\partial x^\lambda}, \\ &P_0^{0} \equiv 1, \quad P_{0'}^{\nu} \equiv 0, \quad P_{\lambda'}^{\nu} \equiv A_{\lambda'}^{\nu} \equiv \frac{\partial x^{\nu}}{\partial x^{\lambda'}}, \quad P_{\lambda'}^{0} \equiv \frac{1}{c(n+1)} A_{\lambda'}^{\nu} \frac{\partial \log \Delta_{(\epsilon)}}{\partial x^\nu} \end{aligned} \tag{3.1}$$

where c is any real number $\neq 0$.

Any tensor, defined in the usual way by means of $A_\lambda^{\nu'}$, $A_{\lambda'}^{\nu}$ only, will be referred to as an *affine* tensor. Any tensor defined by means of the whole set $P_\Sigma^{\Omega'}$, $P_{\Sigma'}^{\Omega}$[7] will be referred to as a *projective* tensor. Especially the projective contravariant vector will be called *a point* and the projective covariant vector will be termed *a hyperplane*. Owing to the special type of $P_\Sigma^{\Omega'}$ and $P_{\Sigma'}^{\Omega}$ the components of a point p^Ω and those of a hyperplane d_Σ transform in the following manner:

$$\begin{aligned} &p^{\nu'} = A_\lambda^{\nu'} p^\lambda, && d_{\lambda'} = A_{\lambda'}^{\nu} \left(d_\nu + \frac{d_0}{c(n+1)} \frac{\partial \log \Delta_{(\epsilon)}}{\partial x^\nu} \right), \\ &p^{0'} = p^0 - \frac{1}{c(n+1)} p^\lambda \frac{\partial \log \Delta_{(\epsilon)}}{\partial x^\lambda}, && d_{0'} = d_0. \end{aligned} \tag{3.2}$$

[5] k in (2.4) is any real number and not an index.

[6] T. Y. Thomas, Math. Ztschr. *25*, pp. 723-733, (1926) for $c = -\frac{1}{n+1}$. Other literature is mentioned in Hlavatý, Annali di Math. *12*, pp. 217-294, (1933-34).

[7] From the transformation law for d_Σ it follows that any hyperplane d_Σ with $d_0 = 0$ is a covariant affine vector.

Among all point fields of our space there is one, E^Ω, whose components in any coördinate system are

$$E^\Omega \equiv C\delta_0^\Omega, \tag{3.3}$$

C being any function $\neq 0$ of the x's. This point field represents the element field $(x^1, x^2, \cdots, x^n)$[8] with which we are dealing in our space. Among all hyperplane fields of our space, there is a privileged one, namely the tangent hyperplane field of our $X_{n-1}^{(n)}$ whose components are

$$t_\lambda \text{ given by (2.2), and } t_0 = 0. \tag{3.4}$$

It follows from this that E^Ω and t_Σ are incident,

$$E^\Omega t_\Omega = 0, \tag{3.5a}$$

and furthermore, t_Σ is intrinsic of degree one

$$\bar{l}_\Sigma = t_\Sigma f. \tag{3.5b}$$

The set of functions

$$\begin{aligned} &\text{a)}\quad \Pi_{\lambda\mu}^\nu \equiv \Pi_{\mu\lambda}^\nu \equiv \Gamma_{\lambda\mu}^\nu - \frac{1}{n+1}(\delta_\lambda^\nu \Gamma_{\alpha\mu}^\alpha + \delta_\mu^\nu \Gamma_{\alpha\lambda}^\alpha), \\ &\text{b)}\quad \Pi_{0\Sigma}^\Omega \equiv \Pi_{\Sigma 0}^\Omega \equiv c\delta_\Sigma^\Omega, \\ &\text{c)}\quad \Pi_{\lambda\mu}^0 \equiv \Pi_{\mu\lambda}^0 \equiv \frac{1}{c(n-1)}\left(-\frac{\partial \Pi_{\lambda\mu}^\alpha}{\partial x^\alpha} + \Pi_{\beta\mu}^\alpha \Pi_{\lambda\alpha}^\beta\right), \end{aligned} \tag{3.6}$$

transforms under $x^{\nu'} \to x^\nu$ according to

$$\Pi_{\Sigma'\Lambda'}^{\Omega'} = P_\Gamma^{\Omega'}(P_\Lambda^\Psi P_\Sigma^\Phi \Pi_{\Phi\Psi}^\Gamma + \partial_{\Sigma'} P_\Lambda^\Gamma), \qquad \left(\partial_{\Sigma'} \equiv \delta_\Sigma^{\lambda'} \frac{\partial}{\partial x^{\lambda'}}\right), \tag{3.7}$$

and consequently constitutes a projective connection. Our space of n dimensions endowed with this projective connection will be designated by $\mathfrak{P}_n$. The projective tensor

$$P_{\Gamma\Sigma\Lambda}^\Omega \equiv \partial_\Sigma \Pi_{\Lambda\Gamma}^\Omega - \partial_\Gamma \Pi_{\Lambda\Sigma}^\Omega + \Pi_{\Phi\Sigma}^\Omega \Pi_{\Lambda\Gamma}^\Phi - \Pi_{\Phi\Gamma}^\Omega \Pi_{\Lambda\Sigma}^\Phi, \qquad \left(\partial_\Sigma \equiv \delta_\Sigma^\lambda \frac{\partial}{\partial x^\lambda}\right), \tag{3.8}$$

is the curvature tensor of $\mathfrak{P}_n$. It satisfies the following set of equations

$$P_{\Gamma\Sigma\Lambda}^\Omega = \delta_\Gamma^\gamma \delta_\Sigma^\sigma \delta_\Lambda^\lambda P_{\gamma\sigma\lambda}^\Omega, \tag{3.9a}$$

$$P_{\Omega\Sigma\Lambda}^\Omega = 0, \tag{3.9b}$$

[8] Any hyperplane h_Σ incident with E^Ω, $E^\Omega h_\Omega = 0$, is a covariant affine vector. (See footnote 7).

the last set being only a transcription of (3.6c) and

$$\Pi^{\Omega}_{\Omega\Sigma} = \Pi^{\Omega}_{\Sigma\Omega} = c(n+1)\delta^{0}_{\Sigma}. \tag{3.10}$$ [9]

5. Projective objects in $X^{(n)}_{n-1}$

Let us first introduce the following set of expressions:

$$\begin{aligned} &P^{\dot{0}'}_{\dot{0}} \equiv 1, \qquad P^{a'}_{\dot{0}} \equiv 0, \qquad P^{a'}_{b} \equiv A^{a'}_{b}, \qquad P^{\dot{0}'}_{b} \equiv -\frac{1}{c'n}\,\partial_b \log \Delta_{(\omega)}, \\ &P^{\dot{0}}_{\dot{0}'} \equiv 1, \qquad P^{a}_{\dot{0}'} \equiv 0, \qquad P^{a}_{b'} \equiv A^{a}_{b'}, \qquad P^{\dot{0}}_{b'} \equiv \frac{1}{c'n}\,A^{a}_{b'}\,\partial_a \log \Delta_{(\omega)}, \end{aligned} \tag{5.1}$$

where c' is any constant $\neq 0$. Any tensor defined in the usual way by means of $A^{a'}_{b}$, $A^{a}_{b'}$ only, will be referred to as an *affine* tensor (in $X^{(n)}_{n-1}$). Any tensor defined in the usual way by means of the whole set $P^{A'}_{B}$, $P^{A}_{B'}$,[2] will be referred to as a *projective* tensor (in $X^{(n)}_{n-1}$). Especially the projective contravariant vector in $X^{(n)}_{n-1}$ will be said to be a *point* (in $X^{(n)}_{n-1}$), and the projective covariant vector in $X^{(n)}_{n-1}$ will be called a *plane* (in $X^{(n)}_{n-1}$).

Throughout this paper we shall deal with the so-called fundamental quadratic projective tensor h_{AB} in $X^{(n)}_{n-1}$. In order to define it we must introduce first the projective unit mixed tensor U^{Ω}_{A} by means of the following identities

$$U^{\Omega}_{0} \equiv E^{\Omega},\ U^{\nu}_{a} \equiv u^{\nu}_{a}, \text{ and } U^{0}_{a} \text{ not yet defined.} \tag{5.2}$$

This tensor enables us to construct h_{AB} by putting

$$h_{AB} \equiv U^{\Omega}_{A} U^{\Sigma}_{B} t_{\Sigma,\Omega}, \tag{5.3a}$$

where $t_{\Sigma,\Omega}$ is the covariant derivative of t_{Σ} with respect to (3.6). Because

$$t_{0,\Omega} = \partial_\Omega t_0 - \Pi^{\Sigma}_{0\Omega} t_\Sigma = -ct_\Omega, \qquad t_{\Omega,0} = \partial_0 t_\Omega - \Pi^{\Sigma}_{\Omega 0} t_\Sigma = -ct_\Omega, \tag{5.4}$$

then according to (3.5a) and (5.2), we may write

$$h_{AB} = \delta^{a}_{A}\delta^{b}_{B} h_{ab} = \delta^{a}_{A}\delta^{b}_{B} u^{\omega}_{a} u^{\sigma}_{b} t_{\sigma,\omega} \tag{5.3b}$$ [10]

---

[9] The set of functions $\Pi^{\Sigma}_{\Sigma\mu}$ transforms in the same manner as the set of the last n components of a hyperplane. Owing to the fact that the same constant c which appears in (3.1) also occurs in (3.6b), the above-mentioned transformation law reduces to the transformation law of a covariant affine vector. But from (3.6) we see that this vector $\Pi^{\Sigma}_{\Sigma\mu}$ is equal to zero. On the other hand, $\Pi^{\Sigma}_{\Sigma 0}$ is a scalar equal to $c(n+1)$. The last two statements are condensed in (3.10), which holds in any coördinate system.

[10] It can easily be shown that according to (3.6b) and (5.2) we have

$$h_{ab} = (\partial_a t_\lambda - u^{\mu}_{a}\Pi^{\nu}_{\lambda\mu} t_\nu) U^{\lambda}_{b} = u^{\lambda}_{b}(\partial_a t_\lambda - u^{\mu}_{a}\Pi^{\nu}_{\lambda\mu} t_\nu). \tag{5.3c}$$

in place of (5.3a). It follows from this that in spite of the not yet defined U_a^0 the tensor h_{AB} is well defined. According to the symmetry conditions $\Pi^{\Omega}_{\Sigma A} = \Pi^{\Omega}_{A\Sigma}$ we get from (5.3)

$$h_{AB} - h_{BA} = U_A^{\Omega} U_B^{\Sigma}(t_{\Sigma,\Omega} - t_{\Omega,\Sigma}) \tag{5.3c}$$
$$= \delta_A^a \delta_B^b u_a^{\omega} u_b^{\sigma}(t_{\sigma,\omega} - t_{\omega,\sigma})$$
$$= \delta_A^a \delta_B^b t_{\omega}(\partial_b u_a^{\omega} - \partial_a u_b^{\omega})$$

so that in general $h_{AB} \neq h_{BA}$ if we are dealing with X_{n-1}^n and on the other hand $h_{AB} = h_{BA}$ for X_{n-1}. Throughout this paper we shall deal with that case only in which

$$h \equiv \det. |h_{ab}| \neq 0. \tag{5.5a}$$

It follows from this that there exists a unique contravariant affine tensor h^{ab} in $X_{n-1}^{(n)}$ satisfying the following equations

$$h_{ad}h^{ab} = h_{da}h^{ba} = \delta_d^b. \tag{5.5b}$$

The projective tensor h_{AB} is intrinsic of degree 1,

$$\bar{h}_{AB} = h_{AB}f; \tag{5.6a}$$

and consequently h^{ab} is intrinsic of degree -1,

$$\bar{h}^{ab} = h^{ab}f^{-1}; \tag{5.6b}$$

and furthermore h is intrinsic of degree $n - 1$,

$$\bar{h} = hf^{n-1}. \tag{5.6c}$$

Consequently the scalar function

$$P^2 \equiv P^{\Phi}_{\Omega\Sigma\Psi} P^{\Psi}_{\Lambda\Gamma\Phi} U_a^{\Omega} U_b^{\Sigma} U_c^{\Lambda} U_d^{\Gamma} h^{ac} h^{bd} \tag{5.7}$$
$$\equiv P^{\varphi}_{\omega\sigma\psi} P^{\psi}_{\lambda\gamma\varphi} u_a^{\omega} u_b^{\sigma} u_c^{\lambda} u_d^{\gamma} h^{ac} h^{bd}$$

is intrinsic of degree -2, $\bar{P}^2 = P^2 f^{-2}$. If $P \neq 0$ in our domain, then by definition this domain of the large space $\mathfrak{P}_n$ is a *general domain* in $\mathfrak{P}_n$. If on the contrary $P = 0$ in our domain, then by definition this domain is a *special* one. We shall speak about the *general* or *special* case accordingly.

If there should be any connection in $X_{n-1}^{(n)}$, say Ξ^A_{BC}, then its coefficients must transform according to

$$\Xi^{A'}_{B'C'} = P^{A'}_{D}(P^{E}_{B'} P^{F}_{C'} \Xi^{D}_{EF} + \partial_{C'} P^{D}_{B'}), \qquad (\partial_{C'} = \delta^{d'}_{C'} \partial_{d'}). \tag{5.8a}$$

It follows from this that the requirement

$$\Xi^A_{AB} = \Xi^A_{BA} = c' \delta^0_B n \tag{5.8b}$$

is invariant under (5.1) (compare with (3.10)). We shall use this fact to define U_a^0.

6. Projective unit mixed tensor

In order to define the components U_a^0 of the projective unit mixed tensor U_A^Ω whose remaining components were defined by (5.2), let us first introduce an affine connection in $X_{n-1}^{(n)}$

$$\Psi_{ba}^{d} = h^{ed}(\partial_a u_b^\lambda + \Pi_{a\mu}^\lambda u_a^\mu u_b^\alpha)(\partial_e t_\lambda - \Pi_{\lambda\omega}^\nu t_\nu u_e^\omega) \tag{6.1a}$$

(which is not invariant under the transformation of the x's) and a set of expressions

$$\mu^\nu = -\frac{h^{ab}}{n-1}(\partial_a u_b^\nu + \Pi_{\lambda\mu}^\nu u_a^\mu u_b^\lambda - \Psi_{ba}^d u_d^\nu). \tag{6.1b}$$

If we transform the x's then this set transforms into a set $\mu^{\nu'}$ where

$$\mu^{\nu'} = A_\lambda^{\nu'}(\mu^\lambda + \beta^\lambda) \tag{6.2a}$$

and β^λ is subject to the condition

$$\beta^\lambda t_\lambda = 0. \tag{6.2b}$$

On the other hand we get from (5.2), (5.3b) and (5.5)

$$\begin{aligned}\mu^\nu t_\nu &= -\frac{h^{ab}}{n-1}(\partial_a u_b^\nu + \Pi_{\lambda\mu}^\nu u_a^\lambda u_b^\mu)t_\nu \\ &= \frac{h^{ab}}{n-1}(u_b^\nu \partial_a t_\nu - \Pi_{\lambda\mu}^\nu u_a^\mu u_b^\lambda t_\nu) = \frac{h^{ab}h_{ab}}{n-1} = 1.\end{aligned} \tag{6.3}$$

From (6.2) we see that (6.3) is a scalar equation. According to this equation the determinant

$$M = \begin{vmatrix} u_1^1, \cdots, u_1^n \\ \vdots \quad\quad \vdots \\ u_{n-1}^1, \cdots, u_{n-1}^n \\ \mu^1, \cdots, \mu^n \end{vmatrix} \neq 0. \tag{6.4}$$

The set (6.1b) transforms under (2.3) according to $\bar{\mu}^\nu = (\mu^\nu - h^{dc}u_c^\nu \partial_d \log f)f^{-1}$ and consequently we have from (6.4)

$$\bar{M} = Mf^{-1}. \tag{6.5a}$$

On the other hand we see that μ^ν and consequently also M is a projective invariant. Hence

M, as defined by (6.4), *is projective and intrinsic of degree* -1.

It transforms under (3.1) in the following manner

$$M' = \Delta_{(e)} M \tag{6.5b}$$

and under (5.1) in the following way

$$M' = \Delta_{(i)}^{-1} M. \tag{6.5c}$$

Consequently M is a scalar density of external weight $w_{(e)} = 1$ (as follows from (6.5b)) and internal weight $w_{(i)} = -1$ (as is seen from 6.5c)). Hence

$$H \equiv hM^{n-1} \tag{6.6}$$

is a scalar density of external weight $w_{(e)} = n - 1$ and internal weight

$$w_{(i)} = -(n-1) - 2 = -(n+1),$$

and *is projective and intrinsic of degree* 0. Moreover it is different from zero. Hence

$$H_{;A} \equiv \partial_A H + (n-1)H\Pi^{\Sigma}_{\Sigma\Omega} U^{\Omega}_A - (n+1)\Xi^B_{BA} H, \qquad (\partial_A = \delta^b_A \partial_b) \tag{6.7a}$$

is a plane in $X^{(n)}_{n-1}$ of external weight $w_{(e)} = n - 1$ and internal weight $w_{(i)} = -(n+1)$ and consequently

$$H_A \equiv H^{-1}H_{;A} = \partial_A \log H + (n-1)\Pi^{\Sigma}_{\Sigma\Omega} U^{\Omega}_A - (n+1)\Xi^B_{BA} \tag{6.7b}$$

is a plane in $X^{(n)}_{n-1}$ which has both weights equal to zero. According to (3.3), (3.10), (5.2), and (5.8) we have

$$H_{\dot{0}} = (n+1)[Cc(n-1) - c'n], \quad H_a = \partial_a \log H + (n+1)(n-1)U^0_a c. \tag{6.7c}$$

If we choose the arbitrary function C in such a way that

$$Cc(n-1) = c'n \tag{6.8}$$

(which choice reduces C to a constant) then the requirement $H_A = 0$ is invariant under (5.1). The foregoing equation is equivalent to

$$H_{\dot{0}} = (n+1)[Cc(n-1) - c'n] \equiv 0$$

and

$$U^0_a = -\frac{1}{c(n^2-1)} \partial_a \log H.^{11} \tag{6.9}$$

The projective unit mixed tensor U^{Ω}_A *is thus defined by* (5.2) *and* (6.9).

Another useful notion can be deduced from the scalar function

$$Q \equiv M^2 h^{-1} \neq 0 \tag{6.10a}$$

which is a scalar density of external weight $w_{(e)} = 2$ and internal weight $w_{(i)} = 0$ and is intrinsic of degree $-(n+1)$,

$$\bar{Q} = Qf^{-(n+1)}. \tag{6.10b}$$

---

[11] We could define U^0_a by starting with (6.9) without any notion of the connection Ξ^D_{BA} Then it can easily be shown that U^0_a transforms in the right way, provided (6.8) is satisfied. The right way which we chose above does not require any computation involving the transformation law.

If we are dealing with X_{n-1}^n then we can define the covariant derivative $Q_{,\Omega}$ of Q in $\mathfrak{P}_n$ which is a hyperplane density of weights $w_{(\bullet)} = 2$, $w_{(\circ)} = 0$. Consequently

$$Q_\Omega \equiv \frac{Q_{,\Omega}}{(n+1)Q} = \frac{1}{n+1}\,(\partial_\Omega \log Q + 2\Pi^{\Sigma}_{\Sigma\Omega}) \tag{6.11a}$$

$$= \frac{\partial_\Omega \log Q}{n+1} + 2c\delta^0_\Omega \qquad \left(\partial_\Omega \equiv \delta^\omega_\Omega \frac{\partial}{\partial x^\omega}\right)$$

is a hyperplane of weights $w_{(\bullet)} = w_{(\circ)} = 0$. The foregoing equations are equivalent to

$$Q_0 = 2c, \qquad Q_\omega = \frac{1}{n+1}\,\partial_\omega \log Q. \tag{6.11b}$$

This hyperplane is not intrinsic as follows from (6.10b) and (6.11a),

$$\bar{Q}_\Omega = Q_\Omega - \partial_\Omega \log f. \tag{6.12a}$$

Starting with this hyperplane, we may define its intersection Q_A with t_Ω which is a plane in X_{n-1}^n

$$Q_A = U^\Omega_A Q_\Omega. \tag{6.13a}$$

From this equation we get

$$\begin{aligned} Q_{\dot{0}} &= U^\Omega_{\dot{0}} Q_\Omega = CQ_0 = 2cC \\ Q_a &= U^\Omega_a Q_\Omega = U^\omega_a Q_\omega + U^0_a Q_0 = \frac{\partial_a \log Q}{n+1} - 2\,\frac{\partial_a \log H}{n^2-1} \\ &= \frac{1}{n+1}\Big[\partial_a \log M^2 - \partial_a \log h. \\ &\qquad - \frac{2}{n-1}\,\partial_a \log M^{n-1} - \frac{2}{n-1}\,\partial_a \log h\Big] \\ &= -\frac{1}{n-1}\,\partial_a \log h. \end{aligned} \tag{6.13b}$$

According to (6.12a) the plane Q_A transforms in the following manner

$$\bar{Q}_A = Q_A - \partial_A \log f. \tag{6.12b}$$

Let us emphasize that Q_Ω exists only if we are dealing with the non-holonomic case; meanwhile Q_A as defined by (6.13b) exists also if our $X^{(n)}_{n-1}$ reduces to X_{n-1}.

The following figure shows us the order[12] of objects just defined:

Object	t_Ω	h_{ab} , h^{ab}, h	$U^\Omega_{\dot{0}}$	U^ν_a	U^0_a	Q_Ω	Q_A	
Order	1, (0)	2, (1)	0	1, (0)	3, (2)	(2)	3, (2)	

[12] An object is of order p in the holonomic case if its construction requires at most the p-th derivatives with regard to the y's. An object is of order p in the non-holonomic case if its construction requires at most the p-th derivatives with regard to the x's. The order in the non-holonomic case will be always placed in brackets.

CHAPTER II

7. Correlation in the non-holonomic case

In the next four paragraphs we shall assume that our hypersurface is non-holonomic, X_{n-1}^n. Then we may define the projective tensor

$$T_{\Omega\Sigma} = t_{\Sigma,\Omega} + Q_\Omega t_\Sigma, \tag{7.1}$$

where $(\ ),_\Omega$ stands for the covariant derivative of $(\ )$ in $\mathfrak{P}_n$. $T_{\Omega\Sigma}$ is intrinsic of degree 1

$$\bar{T}_{\Omega\Sigma} = \bar{t}_{\Sigma,\Omega} + \bar{Q}_\Omega \bar{t}_\Sigma = T_{\Omega\Sigma} f. \tag{7.2}$$

Moreover $T_{\Omega\Sigma}$ is not symmetric in Ω and Σ, for it follows from (3.4), (5.4), (6.11b) and (7.1) that

$$\begin{aligned} T_{\Omega 0} &= t_{0,\Omega} + Q_\Omega t_0 = -ct_\Omega, \\ T_{0\Sigma} &= t_{\Sigma,0} + Q_0 t_\Sigma = -ct_\Sigma + 2ct_\Sigma = ct_\Sigma, \end{aligned} \tag{7.3}$$

so that at least $T_{0\mu} \neq T_{\mu 0}$. In order to find the rank of the determinant T of $T_{\Omega\Sigma}$ let $U_{\dot{n}}^\Omega$ denote any point normalized by the condition $U_{\dot{n}}^\Omega t_\Omega = 1$ and consequently not incident with t_Ω, and furthermore let us introduce the quantities

$$\tau_{\mathfrak{a}\mathfrak{e}} = U_{\mathfrak{a}}^\Omega U_{\mathfrak{e}}^\Sigma T_{\Omega\Sigma}, \qquad (\mathfrak{a}, \mathfrak{e} = \dot{0}, \dot{1}, \cdots, \dot{n}). \tag{7.4}$$

The points $\underset{A}{U}{}^\Omega \equiv U_A^\Omega$ and $U_{\dot{n}}^\Omega$ being linearly independent, the rank of $\tau = \det |\ \tau_{\mathfrak{a}\mathfrak{e}}\ |$ is obviously the same as the rank of T. But since

$$\begin{aligned} \tau_{AB} &= U_A^\Omega U_B^\Sigma T_{\Omega\Sigma} = U_A^\Omega U_B^\Sigma t_{\Sigma,\Omega} = h_{AB}, \\ \tau_{\dot{0}\mathfrak{a}} &= U_{\dot{0}}^\Omega U_{\mathfrak{a}}^\Sigma T_{\Omega\Sigma} = CU_{\mathfrak{a}}^\Sigma T_{0\Sigma} = CcU_{\mathfrak{a}}^\Sigma t_\Sigma = Cc\delta_{\mathfrak{a}}^{\dot{n}}, \\ \tau_{\mathfrak{a}\dot{0}} &= U_{\mathfrak{a}}^\Omega U_{\dot{0}}^\Sigma T_{\Omega\Sigma} = CU_{\mathfrak{a}}^\Omega T_{\Omega 0} = -CcU_{\mathfrak{a}}^\Omega t_\Omega = -Cc\delta_{\mathfrak{a}}^{\dot{n}}, \end{aligned} \tag{7.5}$$

we get

$$\tau = \begin{vmatrix} 0 & 0 & \cdots & 0 & Cc \\ 0 & h_{\dot{1}\dot{1}} & \cdots & h_{\dot{1}\dot{n}-1} & \tau_{\dot{1}\dot{n}} \\ \vdots & \vdots & & & \\ 0 & h_{\dot{n}-1\dot{1}} & \cdots & h_{\dot{n}-1\dot{n}-1} & \tau_{\dot{n}-1\dot{n}} \\ -Cc & \tau_{\dot{n}\dot{1}} & \cdots & \tau_{\dot{n}\dot{n}-1} & \tau_{\dot{n}\dot{n}} \end{vmatrix} = hC^2c^2 \neq 0, \tag{7.6}$$

and consequently also the determinant $T \neq 0$. It follows that there exists a unique contravariant tensor $T^{\Omega\Sigma}$ satisfying the equations

$$T^{\Omega\Sigma} T_{\Omega\Lambda} = T^{\Sigma\Omega} T_{\Lambda\Omega} = \delta_\Lambda^\Sigma. \tag{7.7}$$

By means of $T_{\Omega\Sigma}$ we may associate with any point P^Ω two hyperplanes, namely

$$\underset{1}{\rho P}_\Omega = T_{\Omega\Sigma} P^\Sigma, \qquad \underset{2}{\sigma P}_\Omega = T_{\Sigma\Omega} P^\Sigma; \qquad (\rho, \sigma \text{ any functions} \neq 0). \tag{7.8a}$$

Because $T_{\Omega\Sigma} \neq T_{\Sigma\Omega}$ we have in general ${}^*\rho \underset{1}{P}_\Omega \neq {}^*\sigma \underset{2}{P}_\Omega$ for any ${}^*\rho$, ${}^*\sigma$. By means of $T^{\Omega\Sigma}$ we may associate with any hyperplane R_Ω two points namely

$$\lambda \underset{2}{R}^\Sigma = T^{\Omega\Sigma} R_\Omega, \qquad \mu \underset{1}{R}^\Sigma = T^{\Sigma\Omega} R_\Omega; \qquad (\lambda, \mu \text{ any functions} \neq 0). \tag{7.8b}$$

Because $T^{\Omega\Sigma} \neq T^{\Sigma\Omega}$ we have in general ${}^*\lambda \underset{1}{R}^\Sigma \neq {}^*\mu \underset{2}{R}^\Sigma$ for any ${}^*\lambda$, ${}^*\mu$. The equations (7.8) define a *correlation* which is *not* a polarity because $T_{\Omega\Sigma} \neq T_{\Sigma\Omega}$. For this reason we shall refer to $T_{\Omega\Sigma}$ as a *correlation*. It is of rank $n+1$ (i.e. $T \equiv \det | T_{\Omega\Sigma} | \neq 0$); hence not singular. If $\underset{1}{P}_\Omega$ is incident with P^Σ then $\underset{2}{P}_\Omega$ also is incident with P^Σ as follows from (7.8). The geometrical locus of such point is obviously the quadric

$$Q_{\Omega\Sigma} P^\Omega P^\Sigma = 0, \qquad [Q_{\Omega\Sigma} = \tfrac{1}{2}(T_{\Omega\Sigma} + T_{\Sigma\Omega})]. \tag{7.9}$$

This is one of the two fundamental quadrics of our correlation (the other being $Q^{\Omega\Sigma} R_\Omega R_\Sigma = 0$, where $Q^{\Omega\Sigma} = \frac{1}{2}(T^{\Omega\Sigma} + T^{\Sigma\Omega})$). Because

$$\Omega_{\dot{0}a} \equiv \tfrac{1}{2}(\tau_{\dot{0}a} + \tau_{a\dot{0}}) = \Omega_{a\dot{0}} = 0, \tag{7.10}$$

as follows from (7.5), the determinant

$$\Omega \equiv \det | \Omega_{a\dot{e}} |, \qquad \text{where } \Omega_{a\dot{e}} \equiv U^\Omega_a U^\Sigma_{\dot{e}} Q_{\Omega\Sigma}$$

is

$$\Omega \equiv \begin{vmatrix} 0, & 0 & , \cdots , & 0 & , & 0 \\ 0, & k_{i\dot{i}} & , \cdots , & k_{i\dot{n}-1} & , & \Omega_{i\dot{n}} \\ \vdots & \vdots & & \vdots & & \vdots \\ 0, & k_{\dot{n}-1\dot{i}} & , \cdots , & k_{\dot{n}-1\dot{n}-1} & , & \Omega_{\dot{n}-1\dot{n}} \\ 0, & \Omega_{\dot{n}1} & , \cdots , & \Omega_{\dot{n}\dot{n}-1} & , & \Omega_{\dot{n}\dot{n}} \end{vmatrix} = 0, \qquad [k_{ab} = \tfrac{1}{2}(h_{ab} + h_{ba})], \tag{7.11}$$

and consequently the fundamental quadric (7.9) is degenerate.—Recapitulating our results we have the following theorem:

With any point of our X^n_{n-1} there is associated a correlation of rank $n+1$ which is not a polarity. Its coefficients $T_{\Omega\Sigma}$ given by (7.1), *are intrinsic of degree* 1 *and of order* (2). *One of the fundamental quadrics of this polarity is a degenerate one.*

If there exists a ${}^*\rho$ and ${}^*\sigma$ such that the hyperplanes given by (7.8a) are identical (${}^*\rho \underset{1}{P}_\Omega = {}^*\sigma \underset{2}{P}_\Omega$) then we say *by definition* that the point P^Σ and the corresponding hyperplane are *in involution.* We shall see that E^Ω and t_Ω are in involution. Indeed we have, according to (7.3),

$$T_{\Omega\Sigma} E^\Omega = CT_{0\Sigma} = Cct_\Sigma, \qquad T_{\Sigma\Omega} E^\Omega = CT_{\Sigma 0} = -Cct_\Sigma, \tag{7.12}$$

which proves our statement. Hence

The point E^Ω and the hyperplane t_Ω are in involution with regard to our correlation.

The following theorem can be easily proved:

In order that a point P^A in X^n_{n-1} be incident with its corresponding hyperplanes, it is necessary and sufficient that it lie on the asymptotic cone.

The proof follows from the equation of the asymptotic cone, namely

$$h_{ab}P^aP^b = 0,$$

and from the formula

$$h_{AB} = U^\Omega_A U^\Sigma_B T_{\Omega\Sigma}. \tag{7.13}$$

8. Projectivity in the non-holonomic case

Writing

$$T_\Omega{}^\Sigma = T_{\Omega\Lambda}T^{\Lambda\Sigma}, \qquad T^\Sigma{}_\Omega = T^{\Sigma\Lambda}T_{\Lambda\Omega}, \tag{8.1}$$

we get two new projective tensors related by

$$\begin{aligned} T_\Omega{}^\Lambda T^\Sigma{}_\Lambda &= T_{\Omega\Gamma}T^{\Gamma\Lambda}T^{\Sigma\Psi}T_{\Psi\Lambda} = T_{\Omega\Gamma}T^{\Sigma\Gamma} = \delta^\Sigma_\Omega, \\ T_\Lambda{}^\Sigma T^\Lambda{}_\Omega &= T_{\Lambda\Gamma}T^{\Gamma\Sigma}T^{\Lambda\Psi}T_{\Psi\Omega} = T^{\Gamma\Sigma}T_{\Gamma\Omega} = \delta^\Sigma_\Omega. \end{aligned} \tag{8.2}$$

Because $T_{\Omega\Sigma}$ is not symmetric in $\Omega\Sigma$ these tensors $T_\Omega{}^\Sigma$, $T^\Sigma{}_\Omega$ are not proportional to δ^Σ_Ω. Moreover, the determinant of $T_{\Omega\Sigma}$ and the determinant of $T^{\Omega\Sigma}$ being different from zero, the determinants of $T_\Omega{}^\Sigma$ and $T^\Sigma{}_\Omega$ are also different from zero.

These projective tensors occur in the equations which we get easily from (7.8), namely

$$\begin{aligned} &\text{a)} \quad \varphi \underset{2}{R^\Sigma} = T_\Omega{}^\Sigma \underset{1}{R^\Omega}, && \text{b)} \quad \kappa \underset{1}{R^\Sigma} = T^\Sigma{}_\Omega \underset{2}{R^\Omega}, \\ &\qquad \nu \underset{2}{P_\Omega} = T^\Sigma{}_\Omega \underset{1}{P_\Sigma}, && \qquad \psi \underset{1}{P_\Omega} = T_\Omega{}^\Sigma \underset{2}{P_\Sigma}, \end{aligned} \tag{8.3}$$

where $\varphi, \psi, \kappa, \nu$ are arbitrary fuctions $\neq 0$. Hence the correlation (7.8) induces a projectivity (8.3 a) [and its inverse (8.3 b)] of rank $n+1$. We will only consider the case in which the characteristic matrix of this projectivity has simple elementary divisors. Hence there are at least $n+1$ linearly independent self-conjugate points with regard to this projectivity. From the manner in which this projectivity was obtained it is clear that any self-conjugate point R^Ω is in involution with both coincident hyperplanes which correspond to R^Ω in the correlation, and, conversely, any such hyperplane is self-conjugate with regard to the projectivity. Thus the point E^Ω and the hyperplane t_Σ being in involution, they must be self-conjugate by (8.3). Indeed, we get first from (7.12)

$$T_{\Omega\Lambda}E^\Omega T^{\Sigma\Lambda} = E^\Sigma = Cct_\Lambda T^{\Sigma\Lambda}, \qquad T_{\Lambda\Omega}E^\Omega T^{\Lambda\Sigma} = E^\Sigma = -Cct_\Lambda T^{\Lambda\Sigma}, \tag{8.4}$$

and consequently, according to (7.12)

$$\begin{aligned} E^\Omega T_\Omega{}^\Sigma &= E^\Omega T_{\Omega\Lambda}T^{\Lambda\Sigma} = Cct_\Lambda T^{\Lambda\Sigma} = -E^\Sigma, \\ t_\Lambda T_\Sigma{}^\Lambda &= t_\Lambda T^{\Omega\Lambda}T_{\Sigma\Omega} = \frac{1}{Cc}E^\Omega T_{\Sigma\Omega} = -t_\Sigma, \end{aligned} \tag{8.5}$$

and these equations prove our statement. It follows from this that at least n linearly independent self-conjugate points must be incident with t_Ω and consequently at least one of the $n+1$ self-conjugate points must not be incident with t_Ω. Let $\underset{A}{U^\Omega} \equiv U_A^\Omega$ be the linearly independent self-conjugate points incident with t_Ω and $U_{\dot n}^\Omega$ a self-conjugate point not incident with t_Ω and normalized by the condition $U_{\dot n}^\Omega t_\Omega = 1$. The points $U_\mathfrak{a}^\Omega$ ($\mathfrak{a} = \dot 0, \dot 1, \cdots, \dot n$) being linearly independent, there is a unique set of linearly independent hyperplanes $U_\Omega^\mathfrak{e}$ ($\mathfrak{e} = \dot 0, \dot 1, \cdots, \dot n$) satisfying the equations

$$U_\mathfrak{a}^\Omega U_\Omega^\mathfrak{e} = \delta_\mathfrak{a}^\mathfrak{e}. \qquad (\mathfrak{a}, \mathfrak{e} = \dot 0, \dot 1, \cdots, \dot n). \tag{8.6}$$

Obviously $U_\Omega^\mathfrak{e}$ are self-conjugate with regard to (8.3) and moreover

$$U_{\dot n}^\Omega = t_\Omega \quad \text{and} \quad U_{\dot 0}^\Omega = E^\Omega, \tag{8.7}$$

It will be convenient to investigate the projectivity (8.3) in the frame $U_\mathfrak{a}^\Omega, U_\Omega^\mathfrak{e}$ where its coefficients are

$$\tau_\mathfrak{a}{}^\mathfrak{e} = U_\mathfrak{a}^\Omega U_\Sigma^\mathfrak{e} T_\Omega{}^\Sigma, \qquad \tau^\mathfrak{e}{}_\mathfrak{a} = U_\Sigma^\mathfrak{e} U_\mathfrak{a}^\Omega T^\Sigma{}_\Omega \tag{8.8a}$$

and any point R^Σ has the components

$$\rho^\mathfrak{a} = R^\Omega U_\Omega^\mathfrak{a}, \qquad (R^\Sigma = \rho^\mathfrak{a} U_\mathfrak{a}^\Sigma). \tag{8.8b}$$

Consequently the projectivity (8.3a) may be written in the following way:

$$\underset{2}{\varphi\rho}{}^\mathfrak{e} = \tau_\mathfrak{a}{}^\mathfrak{e} \underset{1}{\rho}{}^\mathfrak{e}. \tag{8.3c}$$

Because the elements $U_\mathfrak{a}^\Omega, U_\Omega^\mathfrak{e}$ of our frame are self-conjugate, it follows that the coefficients $\tau_\mathfrak{a}{}^\mathfrak{e}$ reduce to

$$\tau_\mathfrak{a}{}^\mathfrak{e} = 0 \text{ for } \mathfrak{a} \neq \mathfrak{e}, \qquad \tau_\mathfrak{a}{}^\mathfrak{a} \equiv \tau^\mathfrak{a} \neq 0 \text{ (not summed)}. \tag{8.9a}$$

On the other hand, we get from (8.5), (8.6) and (8.7)

$$\begin{aligned} \tau_{\dot 0}{}^{\dot n} &= U_{\dot 0}^\Omega U_\Sigma^{\dot n} T_\Omega{}^\Sigma = E^\Omega T_\Omega{}^\Sigma U_\Sigma^{\dot n} = -E^\Sigma U_\Sigma^{\dot n} = -\delta_{\dot 0}^{\dot n}, \\ \tau_\mathfrak{a}{}^{\dot n} &= U_\mathfrak{a}^\Omega U_\Sigma^{\dot n} T_\Omega{}^\Sigma = t_\Sigma T_\Omega{}^\Sigma U_\mathfrak{a}^\Omega = -t_\Omega U_\mathfrak{a}^\Omega = -\delta_\mathfrak{a}^{\dot n}. \end{aligned} \tag{8.9b}$$

The equations (8.9) reduce (8.3c) to

$$\underset{2}{\varphi\rho}{}^{\dot 0} = -\underset{1}{\rho}{}^{\dot 0} \qquad \underset{2}{\varphi\rho}{}^\mathfrak{a} = \tau^\mathfrak{a} \underset{1}{\rho}{}^\mathfrak{a} \qquad \underset{2}{\varphi\rho}{}^{\dot n} = -\underset{1}{\rho}{}^{\dot n} \quad \text{(not summed)}. \tag{8.3d}$$

From these expressions it follows that any point with the coördinates

$$\rho^\mathfrak{a} = 0 \tag{8.10a}$$

is preserved (self-conjugate) by (8.3). The components R^Σ of (8.10a) are

$$R^\Sigma = U_{\dot 0}^\Omega \rho^{\dot 0} + U_{\dot n}^\Omega \rho^{\dot n},$$

or

$$R^{\Sigma} = \alpha E^{\Sigma} + r^{\Sigma}, \tag{8.10b}$$ [13]

where $\alpha \equiv \rho^0$ is an arbitrary function and $r^{\Sigma} = U^{\Sigma}_{\dot{n}}\rho^{\dot{n}}$ is one of the preserved points. Obviously r^{Σ} is not incident with t_{Σ} (because $r^{\Omega}t_{\Omega} \neq 0$). Hence we have the following theorem:

The projectivity (8.3) *induced by the correlation is of rank* $n+1$ *and its coefficients, given by* (8.1), *are of order two and intrinsic of degree* 0. *It preserves point by point at least one direction through* E^{Ω} *not incident with* t_{Ω}. *This direction is of order two.*

9. Continuation

The question arises if there are other points not incident with t_{Ω} and not contained in the set (8.10b) which are preserved by the projectivity. In order to investigate this question let us suppose that there is a point X^{Ω} preserved by (8.3), which is not incident with t_{Ω} and not contained in the set (8.10b). Because our projectivity is not the identity the point X^{Ω} must be incident with one of the hyperplanes $U^{\dot{1}}_{\Omega}, \cdots, U^{\dot{n}-\dot{1}}_{\Omega}$, say $U^{\dot{1}}_{\Omega}$.[14] But there are at least n linearly independent self-conjugate points in $U^{\dot{1}}_{\Omega}$ namely $U^{\Omega}_{0}, U^{\Omega}_{\dot{2}}, U^{\Omega}_{\dot{3}}, \cdots, \cdots, U^{\Omega}_{\dot{n}}$. In the same manner as used in the foregoing footnote, we can easily show that the projectivity in $U^{\dot{1}}_{\Omega}$ must be an identity[15] and consequently any point B^{Ω} incident with $U^{\dot{1}}_{\Omega}$ is self-conjugate by (8.3). Let B^{Ω} be such a point which is also incident with t_{Ω} i.e.

$$B^{\Omega}t_{\Omega} = B^{\Omega}U^{\dot{n}}_{\Omega} = 0, \tag{9.1}$$

but *not* coincident with E^{Ω}. Because E^{Ω} is also self-conjugate and contained in $U^{\dot{1}}_{\Omega}$ where the projectivity reduces to identity, then any point

$$Y^{\Omega} = \lambda E^{\Omega} + \mu B^{\Omega} \tag{9.2}$$

---

[13] In order to get $r^{\Sigma}(\neq E^{\Sigma})$ one has to calculate $r^{\nu} \neq 0$ from

$$\lambda r^{\nu} = T^{\nu}_{\Omega}r^{\Omega} \equiv T^{\nu}_{\lambda}r^{\lambda}, \qquad T^{0}_{\lambda}r^{\lambda} = 0,$$

where $\lambda = -1$ is the double root of $\det \mid \lambda\delta^{\Sigma}_{\Omega} - T^{\Sigma}_{\Omega} \mid = 0$.

[14] X^{Ω} is not incident with $U^{\dot{n}}_{\Omega}$ according to our assumption, and it cannot be incident with $U^{\dot{0}}_{\Omega}$ as can be easily shown: In $U^{\dot{0}}_{\Omega}$ there are n linearly independent self-conjugates point $U^{\Omega}_{\dot{1}}, U^{\Omega}_{\dot{2}}, \cdots, U^{\Omega}_{\dot{n}}$, the first $n-1$ being incident with $t_{\Omega} \equiv U^{\dot{n}}_{\Omega}$ and the last one contained in the set (8.10b). If X^{Ω} should be in $U^{\dot{0}}_{\Omega}$, then according to our assumption X^{Ω} must not coincide with any of these points, and consequently we should have in $U^{\dot{0}}_{\Omega}$ $n+1$ self-conjugate points, namely $U^{\Omega}_{\dot{1}}, U^{\Omega}_{\dot{2}}, \cdots, U^{\Omega}_{\dot{n}-\dot{1}}$ incident with t_{Ω}, and $U^{\Omega}_{\dot{n}}$ and $X^{\Omega} \neq U^{\Omega}_{\dot{n}}$ not incident with t_{Ω}. Consequently the projectivity in $U^{\dot{0}}_{\Omega}$ would be an identity. From this and from the last theorem it follows that the projectivity (8.3) would be the identity which is in contradiction with our statements.

[15] It does not follow from this statement that the projectivity (8.3) is an identity, because $U^{\dot{1}}_{\Omega}$ contains the set (8.10b).

is self-conjugate, λ and μ being any functions whatever not both equal to zero. From this statement it follows that both hyperplanes corresponding to Y^Ω by the correlation must be coincident. Those hyperplanes are, according to (7.8a) and (7.12)

$$T_{\Sigma\Omega}Y^\Omega = (\lambda E^\Omega + \mu B^\Omega)T_{\Sigma\Omega} = -\lambda Cct_\Sigma + \mu B^\Omega T_{\Sigma\Omega},$$
$$T_{\Omega\Sigma}Y^\Omega = (\lambda E^\Omega + \mu B^\Omega)T_{\Omega\Sigma} = \lambda Cct_\Sigma + \mu B^\Omega T_{\Omega\Sigma}.$$

Hence there must be some function ξ for which

$$\xi T_{\Omega\Sigma}Y^\Omega = T_{\Sigma\Omega}Y^\Omega$$

and this equation is equivalent to

$$\lambda Cct_\Sigma(1 + \xi) + \mu B^\Omega(\xi T_{\Omega\Sigma} - T_{\Sigma\Omega}) = 0 \tag{9.3a}$$

which must be an identity with regard to λ and μ. From this it follows first that $\xi = -1$ so that (9.3a) reduces to

$$B^\Omega Q_{\Omega\Sigma} = 0, \quad \text{where } Q_{\Omega\Sigma} \equiv \tfrac{1}{2}(T_{\Omega\Sigma} + T_{\Sigma\Omega}). \tag{9.3b}$$

Using the frame U^Ω_a, U^ϵ_Ω we may write for (9.3b) the equivalent equation

$$\beta^\epsilon \Omega_{a\epsilon} = 0, \qquad (a, \epsilon = \dot{0}, \dot{1}, \cdots, \dot{n}), \tag{9.3c}$$

where

$$\beta^\epsilon = U^\epsilon_\Omega B^\Omega, \qquad \Omega_{a\epsilon} = U^\Omega_a U^\Sigma_\epsilon Q_{\Omega\Sigma}.$$

According to (9.1) we have $\beta^{\dot{n}} = 0$ and moreover we have $\Omega_{\dot{0}a} = \Omega_{a\dot{0}} = 0$ (see (7.10)), and

$$\Omega_{ab} = U^\Omega_a U^\Sigma_b Q_{\Omega\Sigma} = \tfrac{1}{2}(h_{ab} + h_{ba}) \equiv k_{ab}.$$

This reduces (9.3c) finally to $\Omega_{\dot{n}b}\beta^b = 0$ and

$$\beta^b k_{ab} = 0. \tag{9.3d}$$

If the asymptotic cone is not degenerate (which we shall always assume), then the determinant of k_{ab} is different from zero. Consequently there is only one solution of (9.3d), namely $\beta^b = 0$ so that

$$B^\Omega = U^\Omega_{\dot{0}}\beta^{\dot{0}}$$

i.e., B^Ω is coincident with E^Ω, which is in contradiction to our assumption. It follows from this that the assumption of the existence of the point X^Ω leads to a contradiction Hence: *The projectivity* (8.3) *induced by the correlation preserves point by point the set* (8.10b), *and if the asymptotic cone is not degenerate, then this is a unique set of points preserved by* (8.3) *which are not* (*with the exception of* E^Ω) *incident with* t_Ω. *This set of order two constitutes the field of projective "normal directions" to* X^n_{n-1}.

10. Canonical plane, normal point

If r^Ω is the point which we have obtained as described in footnote 13, then it is obviously an intrinsic point of degree zero, not incident with t_Ω. It follows from this that

$$n^\Omega = \frac{r^\Omega}{r^\Sigma t_\Sigma} = \frac{r^\Omega}{r^\lambda t_\lambda}$$

is intrinsic of degree -1 and satisfies the "normalizing condition"

$$n^\Omega t_\Omega = 1, \tag{10.1}$$

and so does any point of the set

$$N^\Omega = \rho E^\Omega + n^\Omega, \quad (\rho = \text{any scalar function, intrinsic of degree } -1), \tag{10.2a}$$

constructed by means of n^Ω and belonging to the normal direction

$$N^\Omega t_\Omega = 1, \qquad \bar{N}^\Omega = N^\Omega f^{-1}. \tag{10.2b}$$

The point N^Ω is a self-conjugate one by the projectivity and consequently both the hyperplanes corresponding to it must coincide, i.e.,

$$N^\Omega T_{\Sigma\Omega} \text{ is proportional to } N^\Omega T_{\Omega\Sigma},$$

so that we may restrict ourselves to one only of these expressions, say

$$K_\Sigma = N^\Omega T_{\Sigma\Omega}. \tag{10.3a}$$

Because ρ in (10.2a) is any function whatever, (10.3a) defines a family of hyperplanes, for we have, according to (7.12)

$$K_\Sigma = (\rho E^\Omega + n^\Omega) T_{\Sigma\Omega} = -\rho C c t_\Sigma + n^\Omega T_{\Sigma\Omega}. \tag{10.3b}$$

Because N^Ω is not incident with t_Ω there is no hyperplane in the family (10.3b) which is different from t_Ω and incident with E^Ω. On the other hand, any of the hyperplanes (10.3b) is incident with any point of the plane

$$K_A \equiv K_\Sigma U^\Sigma_A = n^\Omega U^\Sigma_A T_{\Sigma\Omega} = N^\Omega U^\Sigma_A T_{\Sigma\Omega}.^{16} \tag{10.4a}$$

In other words, all the hyperplanes (10.3b) pass through this plane which is the intersection of t_Ω and the family (10.3b). This plane is not incident with E^Ω because, according to (7.12) and (10.2b)

$$K_0 = N^\Omega U^\Sigma_0 T_{\Sigma\Omega} = N^\Omega T_{0\Omega} C = C c N^\Omega t_\Omega = C c \neq 0 \tag{10.4b}$$

and moreover it is intrinsic of degree 0 and of order two.

All the hyperplanes corresponding by the correlation to the points (10.2) *of the normal direction pass through a plane, as given by* (10.3b), *which is called the canonical plane. It is not incident with* E^Ω. *Its components are of order two and intrinsic of degree* 0.

---

[16] The components U^Σ_a are not now restricted to represent the self-conjugate points in t_Ω.

All points of the set (10.2) have the same last n components, namely $N^\nu = n^\nu$. Two points of this set differ only in the 0-components[17] and this difference is $C(\underset{1}{\rho} - \underset{2}{\rho})$ if $\underset{1}{\rho}, \underset{2}{\rho}$ are the corresponding functions in (10.2). We use this fact to select one of the points of the set (10.2) in prescribing the convenient condition which enables us to fix the 0-component of this point. From (10.2a) we have

$$N^\Omega_{,\Omega} = E^\Omega_{,\Omega}\rho + (\partial_\Omega \rho)E^\Omega + n^\Omega_{,\Omega} = \rho E^\Omega_{,\Omega} + n^\Omega_{,\Omega}, \tag{10.5}$$

and consequently the point

$$M^\Omega \equiv \frac{n^\Sigma Q_\Sigma - n^\Sigma_{,\Sigma}}{E^\Lambda_{,\Lambda} - E^\Lambda Q_\Lambda} E^\Omega + n^\Omega = \frac{n^\Sigma Q_\Sigma - n^\Sigma_{,\Sigma}}{c(n-1)} \delta^\Omega_0 + n^\Omega \tag{10.6a}$$

which belongs to the set (10.2a) has the property

$$M^\Omega_{,\Omega} = M^\Omega Q_\Omega. \tag{10.6b}$$

It can be easily shown that M^Ω as given by (10.6a) does not depend on the particular choice of the point n^Ω on the normal direction.[18]

The point M^Ω defined by (10.6) will be referred to as a *normal point.* It is of order 3 and intrinsic of degree -1.

There is a unique point called the normal point on the normal direction, satisfying (10.6b). *Its components are given by* (10.6a) *and are of order* 3 *and intrinsic of degree* -1.

The plane (10.3a), where now $N^\Omega \equiv M^\Omega$ is given by (10.6a), will be referred to as M_Σ,

$$M_\Sigma = M^\Omega T_{\Sigma\Omega}. \tag{10.7a}$$

It is well defined and belongs to the set (10.3b) with the particular value

$$\rho = -\frac{n^\Lambda_{,\Lambda} + n^\Lambda Q_\Lambda}{Cc(n-1)},$$

---

[17] The geometrical meaning of this statement is that two different points of the set (10.2) are linearly dependent upon E^Ω, but incident with two different hyperplanes. This follows from the fact that if S_Ω is not incident with t_Ω (i.e. $E^\Omega S_\Omega \neq 0$) there is only one point in (10.2) which is incident with S_Ω, namely the point N^Ω for which $N^\Omega S_\Omega = 0$ i.e.

$$N^\Omega = -\frac{\dot{n}^\Sigma S_\Sigma}{S_0}\delta^\Omega_0 + n^\Omega.$$

[18] If ${}^*n^\Omega = \sigma E^\Omega + n^\Omega$ is a point on the normal direction and

$${}^*M^\Omega = \frac{{}^*n^\Sigma Q_\Sigma - {}^*n^\Sigma_{,\Sigma}}{c(n-1)}\delta^\Omega_0 + {}^*n^\Omega,$$

then one gets immediately

$${}^*M^\Omega = \frac{Q_\Sigma(\sigma E^\Sigma + n^\Sigma) - \sigma E^\Sigma_{,\Sigma} - n^\Sigma_{,\Sigma}}{c(n-1)}\delta^\Omega_0 + \sigma E^\Omega + n^\Omega = M^\Omega.$$

i.e.

$$M_\Sigma = \frac{n^\Lambda_{,\Lambda} - n^\Lambda Q_\Lambda}{n-1} t_\Sigma + n^\Omega T_{\Sigma\Omega}. \tag{10.7b}$$

Especially we have

$$M_0 = n^\Omega T_{0\Omega} = M^\Omega T_{0\Omega} = cM^\Omega t_\Omega = c \neq 0. \tag{10.7c}$$

This hyperplane is of order 3 and intrinsic of degree 0. Of course its intersection with t_Ω, namely $M_\Sigma U^\Sigma_A$ is again the canonical plane

$$K_A = M_\Sigma U^\Sigma_A. \tag{10.7d}$$

Later we shall use the components $h^{\dot{0}a}$, $h^{a\dot{0}}$ and $h^{\dot{0}\dot{0}}$ which are not yet defined. But using K_A and writing

$$h^{BA}K_A = 0, \qquad h^{0a} = h^{a\dot{0}}, \tag{10.8a}$$

we easily get

$$\begin{aligned} h^{b0} = h^{\dot{0}b} &= -\frac{h^{ba}K_a}{K_{\dot{0}}} = -\frac{h^{ba}K_a}{Cc}, \\ h^{\dot{0}\dot{0}} &= \frac{h^{ba}K_aK_b}{(K_{\dot{0}})^2} = \frac{h^{ba}K_aK_b}{C^2c^2}. \end{aligned} \tag{10.8b}$$

Hence $h^{a0} = h^{\dot{0}a}$ and $h^{\dot{0}\dot{0}}$ defined by (10.8) are of order two and intrinsic of order -1, so that we may say h^{AB} is intrinsic of degree -1.

11. Polarities in the general case

Let us now reconsider the general case of a $X^{(n)}_{n-1}$. We shall try to associate with any point E^Ω of $X^{(n)}_{n-1}$ a polarity with the following properties:

a) its rank is $n+1$,

b) by it the hyperplane t_Ω corresponds to E^Ω, and furthermore

c) it induces in t_Ω the polarity with regard to the asymptotic cone.

If $g_{\Omega\Sigma} = g_{\Sigma\Omega}$ are the coefficients of the polarity to be obtained, then by b), $E^\Omega g_{\Sigma\Omega} (= E^\Omega g_{\Omega\Sigma})$ must be proportional to t_Σ and therefore we may set

$$g_{0\Omega} = g_{\Omega 0} = ct_\Omega. \tag{11.1a}$$

On the other hand, if we put

$$k_{AB} = \tfrac{1}{2}(h_{AB} + h_{BA}) \tag{11.2}$$

then we must have by c)

$$g_{\Omega\Sigma} U^\Omega_A U^\Sigma_B = k_{AB}.^{19} \tag{11.1b}$$

---

[19] According to (11.1a), this equation can be written

$$g_{\lambda\mu} U^\lambda_A U^\mu_B = k_{AB}.$$

If ${}^0g_{\Omega\Sigma}$ is any solution whatever of (11.1b) then the most general solution can be put in the following form,

$$g_{\Omega\Sigma} = {}^0g_{\Omega\Sigma} + t_\Omega a_\Sigma + t_\Sigma a_\Omega + (*)t_\Omega t_\Sigma . \tag{11.3}$$

Here (and throughout this paper) (*) stands for any scalar function whatever, intrinsic of degree -1, while a_Σ is any hyperplane which by (11.1a) and (11.3) must satisfy the following condition

$$ct_\Omega = ct_\Omega + t_\Omega a_0$$

and consequently

$$a_0 = 0. \tag{11.4}$$

Hence a_A is a covariant affine vector and the hyperplane a_Σ must be incident with E^Ω.

In order to choose conveniently this arbitrary hyperplane a_Σ let us first introduce the following mixed projective tensors

$$\begin{aligned} g_{A\Omega\Sigma} &\equiv U^\Lambda_A g_{\Omega\Sigma,\Lambda} + Q_A g_{\Omega\Sigma}, \\ {}^0g_{A\Omega\Sigma} &\equiv U^\Lambda_A {}^0g_{\Omega\Sigma,\Lambda} + Q_A {}^0g_{\Omega\Sigma}, \end{aligned} \tag{11.5a}$$

and then the projective tensors

$$g_{ABD} \equiv U^\Omega_B U^\Sigma_D g_{A\Omega\Sigma}, \qquad {}^0g_{ABD} \equiv U^\Omega_B U^\Sigma_D {}^0g_{A\Omega\Sigma}, \tag{11.5b}$$

which are connected by a relation

$$g_{ABD} = {}^0g_{ABD} + h_{AB}a_D + h_{AD}a_B, \qquad (a_D = U^\Omega_D a_\Omega), \tag{11.6}$$

as follows from (11.3) and (5.3a).

First of all, we have, according to (6.13b) and (3.6b)

$$\begin{aligned} g_{\dot 0\Omega\Sigma} &= Cg_{\Omega\Sigma,0} + Q_{\dot 0}g_{\Omega\Sigma} = C(-\Pi^\Lambda_{\Omega 0}g_{\Lambda\Sigma} - \Pi^\Lambda_{\Sigma 0}g_{\Omega\Lambda}) + 2Ccg_{\Omega\Sigma} \\ &= 0 = {}^0g_{\dot 0\Omega\Sigma}, \end{aligned} \tag{11.7a}$$

$$\begin{aligned} g_{A\dot 0D} = g_{AD\dot 0} &= U^\Lambda_A U^\Sigma_D Cg_{0\Sigma,\Lambda} + CQ_A U^\Sigma_D g_{0\Sigma} \\ &= CU^\Lambda_A U^\Sigma_D g_{0\Sigma,\Lambda} \\ &= CU^\Sigma_D(\partial_A g_{0\Sigma} - \Pi^\Omega_{0\Lambda} g_{\Omega\Sigma} U^\Lambda_A - \Pi^\Omega_{\Sigma\Lambda} g_{0\Omega} U^\Lambda_A) \\ &= CcU^\Sigma_D U^\Lambda_A t_{\Sigma,\Lambda} - CcU^\Sigma_D U^\Lambda_A g_{\Lambda\Sigma} \\ &= Cc(h_{AD} - k_{DA}) \\ &= {}^0g_{A\dot 0D} = {}^0g_{AD\dot 0}, \end{aligned}$$

Setting

$$l_{AD} = \tfrac{1}{2}(h_{AD} - h_{DA}),$$

we may write for the foregoing equation

$$g_{A\dot 0D} = g_{AD\dot 0} = {}^0g_{A\dot 0D} = {}^0g_{AD\dot 0} = Ccl_{AD} \tag{11.7b}$$

(and consequently, if our $X_{n-1}^{(n)}$ is a X_{n-1}, we have

$$g_{ABD} = \delta_A^a \delta_B^b \delta_D^d g_{abd}, \qquad {}^0g_{ABD} = \delta_A^a \delta_B^b \delta_D^d {}^0g_{abd}). \tag{11.7c}$$

From (11.4) and (11.6) we have

$$\begin{aligned} g_{ABD}h^{AB} &= {}^0g_{ABD}h^{AB} + h^{AB}h_{AB}a_D + h^{AB}h_{AD}a_B \\ &= {}^0g_{ABD}h^{AB} + na_D.^{20} \end{aligned} \tag{11.8}$$

We want to define the arbitrary plane a_D in such a way that

$$g_{ABD}h^{AB} = K_D l_{AB} h^{AB}, \tag{11.9a}$$

where in the non-holonomic case K_A, the canonical plane, is defined by (10. 4a) while in the holonomic case $l_{AB} = 0$ (and K_A is not defined). We get from (11.8) and (11.9 a)

$$a_D = \frac{1}{n}[K_D l_{AB} - {}^0g_{ABD}]h^{AB}.^{21} \tag{11.10}$$

If we write ${}^0a_\Sigma$ for a solution of

$$a_\Sigma U_D^\Sigma = \frac{1}{n}[K_D l_{AB} - {}^0g_{ABD}]h^{AB} \quad (= a_D),^{22} \tag{11.11a}$$

then the general solution of (11.11a) may be written

$$a_\Sigma = {}^0a_\Sigma + (*)t_\Sigma . \tag{11.11b}$$

Substituting this expression in (11.3) we get

$$g_{\Omega\Sigma} = {}^0g_{\Omega\Sigma} + t_\Omega {}^0a_\Sigma + t_\Sigma {}^0a_\Omega + (*)t_\Omega t_\Sigma , \tag{11.12}$$

where now ${}^0a_\Sigma$ is a solution of (11.11a).

Starting with a particular solution ${}^0g_{\Omega\Sigma}$ of (11.1b) we have (11.12). Let us now consider the influence of this particular choice of ${}^0g_{\Omega\Sigma}$ to the result (11.12). In order to do that, let us start with another particular solution, ${}^0g^*_{\Omega\Sigma}$ of (11.1b). Then ${}^0g_{\Omega\Sigma}$ and ${}^0g^*_{\Omega\Sigma}$ must be related by

$${}^0g^*_{\Omega\Sigma} = {}^0g_{\Omega\Sigma} + t_\Omega X_\Sigma + t_\Sigma X_\Omega + (*)t_\Omega t_\Sigma , \tag{11.13}$$

---

[20] $h^{\dot{0}A} = h^{A\dot{0}}$ is not defined in the case of an X_{n-1}. But in this case g_{ABD}, ${}^0g_{ABD}$ reduce to (11.7c) so that in the holonomic case $h^{\dot{0}A} = h^{A\dot{0}}$ occurs neither in $g_{ABD}h^{AB}$ nor in ${}^0g_{ABD}h^{AB}$.

[21] Using (10.4b) and (11.7b) we have from (11.10)

$$a_{\dot{0}} = \frac{1}{n}[Cc l_{AB} - Cc l_{AB}]h^{AB} = 0,$$

so that $a_\Omega U_{\dot{0}}^\Omega = Ca_{\dot{0}} = a_{\dot{0}} = 0$ (see (11.4)).

[22] If we are dealing with the holonomic case, then (11.11 a) reduces to

$$a_\Sigma U_D^\Sigma = -\frac{1}{n} {}^0g_{abd}h^{ab}\delta_D^d .$$

where X_Ω is a hyperplane incident with E^Ω (i.e. $X_0 = 0$). From this we get first

$$^0g^*_{ABD} = {}^0g_{ABD} + h_{AB}X_D + h_{AD}X_B, \tag{11.14}$$

where

$$X_D = U^\Sigma_D X_\Sigma, \qquad {}^0g^*_{ABD} = U^A_A U^\Omega_B U^\Sigma_D {}^0g^*_{\Omega\Sigma,A} + Q_A k_{BD}$$

and consequently

$$^0g^*_{ABD}h^{AB} = {}^0g_{ABD}h^{AB} + nX_D. \tag{11.15}$$

If we designate by a^*_D the plane defined by means of (11.10) where $^0g_{ABD}$ is to be substituted by $^0g^*_{ABD}$, then we get from (11.10) itself and from (11.15)

$$a^*_D = a_D - X_D. \tag{11.16}$$

Thus, if $^0a^*_\Sigma$ is a solution of

$$a^*_\Sigma U^\Sigma_D = a^*_D, \tag{11.17}$$

we may write

$$^0a^*_\Sigma = {}^0a_\Sigma - X_\Sigma + (*)t_\Sigma, \tag{11.18a}$$

and the most general solution of equation (11.17) may be written

$$a^*_\Sigma = {}^0a^*_\Sigma + (*)t_\Sigma = {}^0a_\Sigma - X_\Sigma + (*)t_\Sigma. \tag{11.18b}$$

On the other hand the projective tensor

$$g^*_{\Omega\Sigma} = {}^0g^*_{\Omega\Sigma} + t_\Omega a^*_\Sigma + t_\Sigma a^*_\Omega + (*)t_\Omega t_\Sigma \tag{11.19}$$

satisfies a condition analogous to (11.9a), namely

$$g^*_{ABD}h^{AB} = K_D l_{AB}h^{AB}. \tag{11.9b}$$

Substituting (11.18b) in (11.19) we get, according to (11.13)

$$\begin{aligned} g^*_{\Omega\Sigma} &= {}^0g_{\Omega\Sigma} + t_\Omega X_\Sigma + t_\Sigma X_\Omega + t_\Omega({}^0a_\Sigma - X_\Sigma) + t_\Sigma({}^0a_\Omega - X_\Omega) + (*)t_\Omega t_\Sigma \\ &= g_{\Omega\Sigma} + (*)t_\Omega t_\Sigma . \end{aligned} \tag{11.20}$$

Hence, choosing arbitrarily the solution $^0g_{\Omega\Sigma}$ we get one of the tensors of the family (11.20) and any tensor of this family satisfies the corresponding condition (11.9b). If we choose a point $U^\Omega_{\dot n}$ (with the normalized condition $U^\Omega_{\dot n}t_\Omega = 1$) in such a way that $U^\Omega_{\mathfrak a}$, ($\mathfrak a = \dot 0, 1, \cdots, \dot n$) are linearly independent, then the rank of the determinant of

$$\gamma^*_{\mathfrak a\mathfrak e} = U^\Omega_{\mathfrak a} U^\Sigma_{\mathfrak e} g^*_{\Omega\Sigma}, \qquad (\mathfrak a, \mathfrak e = \dot 0, \dot 1, \cdots, \dot n),$$

is obviously the same as the rank of the determinant of $g^*_{\Omega\Sigma}$. But since

$$\begin{aligned} \gamma^*_{\mathfrak e\dot 0} &= \gamma^*_{\dot 0\mathfrak e} = U^\Omega_{\dot 0} U^\Sigma_{\mathfrak e} g^*_{\Omega\Sigma} = CU^\Sigma_{\mathfrak e} g^*_{0\Sigma} = C\delta^{\dot n}_{\mathfrak e} = \gamma_{\mathfrak e\dot 0} = \gamma_{\dot 0\mathfrak e}, \\ \gamma^*_{AB} &= \gamma^*_{BA} = U^\Omega_A U^\Sigma_B g^*_{\Omega\Sigma} = k_{AB} = \gamma_{AB} = \gamma_{BA}, \\ \gamma^*_{\dot n\mathfrak a} &= \gamma^*_{\mathfrak a\dot n} = U^\Omega_{\dot n} U^\Sigma_{\mathfrak a} g^*_{\Omega\Sigma} = \gamma_{\dot n\mathfrak a} + (*)\delta^{\dot n}_{\mathfrak a} = \gamma_{\mathfrak a\dot n} + (*)\delta^{\dot n}_{\mathfrak a}, \end{aligned}$$

we have

$$\gamma^* = \begin{vmatrix} \overset{*}{\gamma}_{\dot{0}\dot{0}}, \cdots, \overset{*}{\gamma}_{\dot{0}\dot{n}} \\ \vdots \quad\quad \vdots \\ \overset{*}{\gamma}_{\dot{n}\dot{0}}, \cdots, \overset{*}{\gamma}_{\dot{n}\dot{n}} \end{vmatrix} = \gamma = \begin{vmatrix} \gamma_{\dot{0}\dot{0}}, \cdots, \gamma_{\dot{0}\dot{n}} \\ \vdots \quad\quad \vdots \\ \gamma_{\dot{n}\dot{0}}, \cdots, \gamma_{\dot{n}\dot{n}} \end{vmatrix} = (-1)^{2n+3} C^2 c^2 \det |k_{ab}|.$$

Hence if the asymptotic cone does not degenerate, then $\det |\overset{*}{\gamma}_{\dot{a}\dot{c}}| = \det |\gamma_{\dot{a}\dot{c}}| \neq 0$ and consequently any of the tensors (11.20) has the rank $n+1$. Hence we see that the polarities defined by any of these tensors have the rank $n+1$. From the manner in which the family (11.20) was obtained, it is obvious that any $\overset{*}{g}_{\Omega\Sigma}$, (for an arbitrary function (*)) is of order 3, (2) and intrinsic of degree 1.

Any polarity associated with E^Ω and satisfying the conditions a), b), c) is given by a projective tensor $g_{\Omega\Sigma} = g_{\Sigma\Omega}$ satisfying (11.9). *Any such projective tensor, satisfying* (11.9) *belongs to a family* (11.20) *of tensors. Any tensor of this family is of order* 3, (2) *and intrinsic of degree* 1.

12. The canonical plane of $X_{n-1}^{(n)}$

In the next section we choose one of the polarities (11.20) by prescribing some additional intrinsic and projective conditions. In order to do that we need first some new notions which we introduce in this section. The projective tensor $\overset{*}{g}_{ABD}$ belonging to $\overset{*}{g}_{\Omega\Sigma}$ as given by (11.20) is independent of the particular choice of the function (*) in (11.20)

$$\overset{*}{g}_{ABD} = g_{ABD}. \tag{12.1}$$

This tensor does not vanish identically,[23] and consequently we can restrict ourselves to the case $g_{ABD} \neq 0$. Furthermore, the scalar

$$P \equiv g_{ABD} g_{EFH} h^{AE} h^{BF} h^{DH} = \overset{*}{g}_{ABD} \overset{*}{g}_{EFH} h^{AE} h^{BF} h^{DH} \tag{12.2}$$

[24]

(which in the holonomic case does not involve $h^{\dot{0}A}$) does not vanish identically[25] and consequently we can restrict ourselves to the case $P \neq 0$. P is intrinsic of degree -1 (and does not involve h^{0A} in the holonomic case).

After having defined P, we may define the *canonical plane* K_A of our $X_{n-1}^{(n)}$ in the following way.

1) If our $X_{n-1}^{(n)}$ is a X_{n-1}^{n}, then K_A is defined by (10.4a) and consequently is of order (2).

---

[23] The requirement $g_{ABC} = g_{BAC}$ imposes a special condition on $P^\Omega_{\Gamma\Sigma\Delta}$ which in general is not satisfied. Therefore g_{ABC} does not vanish identically.

[24] We purposely use the same letter P for this scalar as in (5.7).

[25] If, for instance, our subspace is an X_{n-1} and our point E^Ω of X_{n-1} is an elliptic one, then we may choose a coördinate system at this point in which $h^{ab} = \delta^{ab}$ and consequently, according to (11.7c) the scalar P can be expressed in this special coördinate system as follows

$$P = \Sigma (g_{abd})^2 \qquad \text{at } E^\Omega,$$

so that, if $g_{abd} \neq 0$, we have also $P \neq 0$ at E^Ω.

2) If our $X_{n-1}^{(n)}$ is a X_{n-1} then K_A is defined by

$$K_A = \tfrac{1}{2}(Q_A - \partial_A \log P), \tag{12.3}$$

where in the *general* case P is defined by (5.7) and in the *special* case is defined by (12.2). But in any case we have

$$K_{\dot{0}} = Cc,$$

and consequently the canonical plane is not incident with E^Ω. From the manner in which K_A was obtained it is clear that K_A *is intrinsic of degree* 0 *and furthermore of order* 3, 4 (2).[26]

We use K_A for defining the components $h^{\dot{0}A} = h^{A\dot{0}}$ saying that $h^{\dot{0}A} = h^{A\dot{0}}$ *is always defined by* (10.8b) where it must be understood that K_A is defined in the manner just mentioned. It follows from this that $h^{\dot{0}A} = h^{A\dot{0}}$ *are of order* 3, 4, (2) *and intrinsic of degree* -1.

13. The privileged polarity

The projective tensor (11.12) belongs to the family (11.20), where (*) in (11.12) is not related to (*) of (11.20). In this section we shall choose for (*) in (11.12) a function designated by w, and we shall keep the symbol (*) for an arbitrary function in (11.20). Hence we shall write for (11.12) and (11.20)

$$g_{\Omega\Sigma} = {}^0g_{\Omega\Sigma} + t_\Omega {}^0a_\Sigma + t_\Sigma {}^0a_\Omega + wt_\Omega t_\Sigma, \tag{13.1}$$

$$g^*_{\Omega\Sigma} = g_{\Omega\Sigma} + (*)t_\Omega t_\Sigma, \tag{13.2}$$

and prescribe an intrinsic condition for (*), starting with w. In order to do that let us introduce first the mixed intrinsic tensor of degree 1

$$T_{A\Sigma} \equiv U^\Omega_A t_{\Sigma,\Omega} + Q_A t_\Sigma \tag{13.3a}$$

(so that we have in the non-holonomic case, according to (7.1)

$$T_{A\Sigma} = U^\Omega_A T_{\Omega\Sigma}, \tag{13.3b}$$

and in any case

$$U^\Omega_B T_{A\Omega} = h_{AB}). \tag{13.3c}$$

From (13.3a) one easily gets

$$\begin{aligned} T_{\dot{0}\Sigma} &= Ct_{\Sigma,0} + Q_{\dot{0}} t_\Sigma = Cct_\Sigma, \\ T_{A0} &= U^\Omega_A t_{0,\Omega} + Q_A t_0 = 0. \end{aligned} \tag{13.3d}$$

The tensor $T_{A\Sigma}$ appears in the equation which we get easily from (13.1)

$$\begin{aligned} g_{AB\Sigma} &= {}^0g_{AB\Sigma} + a_B T_{A\Sigma} + {}^0a_\Sigma h_{AB} + t_\Sigma(wh_{AB} + U^\Omega_A U^\Lambda_B {}^0a_{\Lambda,\Omega}) \\ &\text{(where } g_{AB\Sigma} = U^\Omega_B g_{A\Omega\Sigma}, \qquad {}^0g_{AB\Sigma} = U^\Omega_B {}^0g_{A\Omega\Sigma}). \end{aligned} \tag{13.4}$$

---

[26] The first number always refers to the *holonomic* subspace in the *general* case, the second number always refers to the *holonomic* subspace in the *special* case, and the number in brackets refers to the *non-holonomic* subspace.

Let us now consider the hyperplane

$$A_\Sigma = ({}^0g_{AB\Sigma} + a_B T_{A\Sigma} + {}^0a_\Sigma h_{AB})h^{AB}. \tag{13.5}$$

Using the foregoing formula (11.10) we get

$$U_D^\Sigma A_\Sigma = {}^0g_{ABD}h^{AB} + na_D = K_D l_{AB}h^{AB}. \tag{13.6}$$

From this it follows that we may choose the hyperplane ${}^0a_\Sigma$ (which is a particular solution of (11.11a)) in such a way that

$$A_\Sigma = M_\Sigma l_{AB}h^{AB}, \tag{13.7}$$

where M_Σ in the non-holonomic case is defined by (10.7b) (and satisfies (10.7c, d)) while in the holonomic case $l_{AB} = 0$. In order that this equation hold it is necessary and sufficient that

$${}^0a_\Sigma = \frac{1}{n-1}[M_\Sigma l_{AB} - {}^0g_{AB\Sigma} - a_B T_{A\Sigma}]h^{AB}. \tag{13.8}$$

Substituting this expression in (13.4) we get

$$g_{AB\Sigma}h^{AB} = [M_\Sigma l_{AB} + t_\Sigma(wh_{AB} + U_A^\Omega U_B^\Lambda {}^0a_{\Lambda,\Omega})]h^{AB}. \tag{13.9}$$

On the other hand, one gets easily from (13.2)

$$g^*_{AB\Sigma} = g_{AB\Sigma} + (*)h_{AB}t_\Sigma \tag{13.10}$$

and consequently, in accordance with (13.9)

$$g^*_{AB\Sigma}h^{AB} = \{t_\Sigma[(w + (*))h_{AB} + U_A^\Omega U_B^\Lambda {}^0a_{\Lambda,\Omega}] + M_\Sigma l_{AB}\}h^{AB}. \tag{13.11}$$

From this equation it follows that we may choose the arbitrary function (*) in such a way that

$$g^*_{AB\Sigma}h^{AB} = M_\Sigma l_{AB}h^{AB}. \tag{13.12}$$

In order that this equation hold it is necessary and sufficient that

$$(*) = -\frac{1}{n-1}U_A^\Omega U_B^\Lambda {}^0a_{\Lambda,\Omega} - w, \tag{13.13}$$

where ${}^0a_\Sigma$ is defined by (13.8). Hence the tensor defined by (13.2) and (13.13) satisfies (13.12).—Let us now consider the influence of the particular choice of w and ${}^0g_{\Omega\Sigma}$ (in (13.1) on this result. It was shown[27] that starting with another tensor ${}^0g_{\Omega\Sigma}$, say ${}^0\tilde{g}$, the corresponding tensor $\tilde{g}_{\Omega\Sigma}$ is related to $g_{\Omega\Sigma}$ by

$$\tilde{g}_{\Omega\Sigma} = g_{\Omega\Sigma} + \tilde{w}t_\Omega t_\Sigma,$$

where now $\tilde{w}$ is a function, which is in general different from w. It follows from this that in studying the question mentioned above, we may confine ourselves to the case in which ${}^0g_{\Omega\Sigma}$ stays fixed, and only w changes. In order to do that

[27] In a slightly different notation; see (11.13)–(11.20).

let us start with another function w and let us denote by the index 1 all the notions belonging to w. First of all we have, in accordance with (13.8) ${}^0_1a_\Sigma = {}^0a_\Sigma$, because this definition does not involve w at all. Then we have from (13.1) and (13.13)

$$(13.14)\qquad (*)_1 = -\frac{1}{n-1}U^\Omega_A U^\Sigma_B {}^0_1a_{\Lambda,\Omega} - w_1 = (*) + w - w_1,$$

$$(13.15)\qquad g_{1\Omega\Sigma} = {}^0_1g_{\Omega\Sigma} + t_\Omega {}^0_1a_\Sigma + t_\Sigma {}^0_1a_\Omega + w_1 t_\Omega t_\Sigma = g_{\Omega\Sigma} + (w - w)_1 t_\Omega t_\Sigma.$$

Substituting from (13.14) and (13.15) in

$$g^*_{1\Omega\Sigma} = g_{1\Omega\Sigma} + (*)_1 t_\Omega t_\Sigma,$$

we get

$$g^*_{1\Omega\Sigma} = g_{\Omega\Sigma} + t_\Omega t_\Sigma (w - w)_1 + t_\Omega t_\Sigma [(*) + w - w]_1 = g^*_{\Omega\Sigma}$$

and consequently the tensor defined by (13.2) and (13.13) which satisfies (13.12) is independent of the particular choice of the tensor (13.1) from the family (11.20). From the manner in which it was obtained it is obvious that it is of order 4, 5, (3) Hence:

In the family (11.20) *there is a unique projective tensor which satisfies* (13.12). *This tensor is defined by* (13.2) *and* (13.13) *where* $g_{\Omega\Sigma}$ *is any tensor of the family* (11.20). *It is of order* 4, 5, (3). *The corresponding polarity, defined by this tensor, will be called the privileged polarity.*

Chapter III

14. Connection in $X^{(n)}_{n-1}$

Let us suppose that we have a connection Ξ^A_{BD} in the $X^{(n)}_{n-1}$. Then (5.8a) must hold and consequently the requirement (5.8b) is invariant under (5.1). Using both connections, namely $\Pi^\Omega_{A\Sigma}$ and Ξ^A_{BC}, we may define the mixed covariant projective derivative $\phi_{;A}$ of any projective tensor ϕ, mixed or not. This construction of $\phi_{;A}$ is the same as in affine case[28] so that we have for instance for $\phi \equiv U^\Omega_B$

$$(14.1a)\qquad U^\Omega_{B;A} \equiv \partial_A U^\Omega_B + \Pi^\Omega_{A\Sigma} U^\Sigma_A U^A_B - \Xi^D_{BA} U^\Omega_D.$$

Setting

$$(14.2)\qquad U^\Omega_{B,A} \equiv \partial_A U^\Omega_B + \Pi^\Omega_{A\Sigma} U^\Sigma_A U^A_B, \qquad H^\Omega_{AB} \equiv U^\Omega_{B;A}$$

we may write for (14.1a)

$$(14.1b)\qquad H^\Omega_{AB} = U^\Omega_{B,A} - \Xi^D_{BA} U^\Omega_D.$$

---

[28] See J. A. Schouten-D. J. Struik "*Einführung . . .*" (Noordhoff, Groningen 1935) pp. 93–97.

H^{Ω}_{AB} is a mixed tensor which satisfies the equation

$$H^{\Omega}_{AB}t_{\Omega} = \underset{B}{U^{\Omega}_{,A}}t_{\Omega} = -U^{\Lambda}_{A}U^{\Omega}_{B}t_{\Omega,\Lambda} = -h_{AB} \tag{14.3}$$

and consequently can be written

$$H^{\Omega}_{AB} = -h_{AB}H^{\Omega} - K^{D}_{AB}U^{\Omega}_{D}, \tag{14.4a}$$

where K^{D}_{AB} is a projective tensor in $X^{(n)}_{n-1}$ and H^{Ω} is a point not incident with t_{Ω} which satisfies the equation

$$H^{\Omega}t_{\Omega} = 1. \tag{14.4b}$$

Hence the equation (14.1b) can be written

$$-h_{AB}H^{\Omega} - K^{D}_{AB}U^{\Omega}_{D} = \underset{B}{U^{\Omega}_{,A}} - \Xi^{D}_{BA}U^{\Omega}_{D}. \tag{14.1c}$$

Setting

$$\Pi^{D}_{BA} \equiv \Xi^{D}_{BA} - K^{D}_{AB} \tag{14.5}$$

we get another connection in $X^{(n)}_{n-1}$ with the coefficients Π^{D}_{BA} and we may write for (14.1c)

$$-h_{AB}H^{\Omega} = \underset{B}{U^{\Omega}_{,A}} - \Pi^{D}_{BA}U^{\Omega}_{D}. \tag{14.6a}$$

Remembering that

$$\underset{B}{U^{\Omega}_{,0}} = \Pi^{\Omega}_{\Lambda\Sigma}U^{\Sigma}_{0}U^{\Lambda}_{B} = CcU^{\Omega}_{B} = \underset{\dot{0}}{U^{\Omega}_{,B}} \tag{14.7}$$

we get from (14.6a) in accordance with (5.3b)

$$\Pi^{D}_{0A} = \Pi^{D}_{A\dot{0}} = Cc\delta^{D}_{A}.^{29,30} \tag{14.6b}$$

If the point H^{Ω} is given, then we may calculate the Π^{D}_{BA} by means of it from (14.6a). In order to do that let us introduce first a new mixed tensor

$$h_{A\Sigma} = T_{A\Sigma} - K_{A}t_{\Sigma}, \tag{14.8a}$$

where $T_{A\Sigma}$ is defined by (13.3a) and K_{A} is the canonical plane, defined in §12. Because we have

$$\begin{aligned} h_{A0} &= T_{A0} - K_{A}t_{0} = 0, \\ h_{\dot{0}\Sigma} &= T_{\dot{0}\Sigma} - K_{\dot{0}}t_{\Sigma} = 0, \end{aligned} \qquad U^{\Sigma}_{B}h_{A\Sigma} = h_{AB}, \tag{14.8b}$$

---

[29] Using the law of transformation for the coefficients of a connection γ^{A}_{BD} which satisfy the equations $\gamma^{A}_{A\dot{0}} = Ccn$ it can easily be proved that γ^{A}_{Ad}, $\frac{1}{n}\gamma^{A}_{A\dot{0}}$ is a plane not incident with E^{Ω}.

[30] From (5.8b), (6.8), (14.5) and (14.6b) we get a condition for K^{D}_{AB}, namely

$$K^{D}_{D\dot{0}} = K^{D}_{\dot{0}D} = -Cc.$$

we get from (14.6a)

$$-h_{AB}H^{\Omega}h_{D\Omega} = -h_{AB}H^{\omega}h_{D\omega} = \underset{B}{U^{\omega}_{,A}}h_{D\omega} - \Pi^{E}_{BA}\,h_{DE}$$

and consequently

$$\Pi^{e}_{BA} = (\underset{B}{U^{\omega}_{,A}} + h_{AB}H^{\omega})h_{D\omega}h^{De} = h_{d\omega}h^{de}(\underset{B}{U^{\omega}_{,A}} + h_{AB}H^{\omega}). \tag{14.9a}$$

On the other hand, we have from (14.6a)

$$-h_{AB}H^{0} = \underset{B}{U^{0}_{,A}} - \Pi^{D}_{BA}U^{0}_{D} = \underset{B}{U^{0}_{,A}} - \Pi^{e}_{BA}U^{0}_{e} - \Pi^{\dot{0}}_{BA}C.$$

Substituting from (14.9a) in this equation we get

$$\Pi^{\dot{0}}_{BA} = \frac{1}{C}[(\underset{B}{U^{\Omega}_{,A}} + h_{AB}H^{\Omega})(\delta^{0}_{\Omega} - U^{0}_{e}\delta^{\omega}_{\Omega}h_{d\omega}h^{de})]. \tag{14.9b}$$

Both the equations (14.9a, b) may be condensed into one, namely

$$\Pi^{D}_{BA} = (\underset{B}{U^{\Omega}_{,A}} + h_{AB}H^{\Omega})\left[\delta^{D}_{e}\delta^{\omega}_{\Omega}h_{d\omega}h^{de} + \frac{\delta^{D}_{0}}{C}(\delta^{0}_{\Omega} - U^{0}_{e}\delta^{\omega}_{\Omega}h_{d\omega}h^{de})\right]. \tag{14.9c}$$

If a connection Π^{D}_{BA} is defined by (14.9) we shall refer to it as a "connection induced by the point H^{Ω}." On the other hand, starting with a given connection Π^{D}_{BA} which is related to H^{Ω} by (14.6 a) we may express H^{Ω} by means of it from (14.6a) and we get

$$H^{\Omega} = -\frac{h^{AB}}{n-1}(\underset{B}{U^{\Omega}_{,A}} - \Pi^{D}_{BA}U^{\Omega}_{D}) \tag{14.10}$$

and this equation reduces according to (14.7) and (14.6b) to

$$H^{\Omega} = -\frac{h^{ab}}{n-1}(\underset{b}{U^{\Omega}_{,a}} - \Pi^{D}_{ba}U^{\Omega}_{D}). \tag{14.11}$$

If a point H^{Ω} is defined by means of (14.11) we shall refer to it as a "point H^{Ω} induced by the connection Π^{D}_{BA}."

If a point H^{Ω} and a connection Π^{D}_{BA} are related by (14.6) we refer to both objects as being "in involution." If H^{Ω} and Π^{D}_{BA} are in involution, then Π^{D}_{BA} is induced by H^{Ω} and satisfies (14.9 c), and conversely H^{Ω} is induced by Π^{D}_{BA} and satisfies (14.11).

15. Normal direction; normal connections

Let us consider the set of equations

$$N^{\Sigma}h_{A\Sigma} = 0, \tag{15.1}$$

which is equivalent to

$$N^{\lambda}h_{a\lambda} = 0, \tag{15.2a}$$

and also to

$$N^\lambda T_{a\lambda} = N^\lambda t_\lambda K_a . \tag{15.2b}$$

If $N^\lambda t_\lambda = 0$ then the plane $T_A \equiv N^\Sigma T_{A\Sigma}$ should have the components

$$T_a \equiv N^\Sigma T_{a\Sigma} \equiv N^\lambda T_{a\lambda} = 0, \qquad T_{\dot\imath} \equiv N^\Sigma T_{\dot\imath\Sigma} \equiv CcN^\lambda t_\lambda = 0. \tag{15.3}$$

But the rank of the determinant

$$t \equiv \begin{vmatrix} T_{i1}, & T_{i2}, & \cdots, & T_{in} \\ \vdots & \vdots & & \\ T_{\dot n-i\,1}, & T_{\dot n-i\,2}, & \cdots, & T_{\dot n-i\,n} \\ t_1, & t_2, & \cdots, & t_n \end{vmatrix} \tag{15.4}$$

being n[31] there is only a unique solution of (15.3), namely $N^\lambda = 0$ and consequently N^Ω coincides with E^Ω. If, on the contrary, $N^\lambda t_\lambda \neq 0$ then we may suppose, without loss of generality

$$N^\lambda t_\lambda = 1 \tag{15.5a}$$

and consequently (15.2b) reduces to

$$N^\lambda T_{a\lambda} = K_a . \tag{15.5b}$$

The determinant of the system (15.5a, b) being $t \neq 0$, this system admits a unique solution $N^\lambda \neq 0$.

It follows from this discussion that the most general solution of (15.1) is

$$*N^\Omega = (*)E^\Omega + \lambda\,{}^0N^\Omega, \qquad (\lambda = \text{any scalar function}), \tag{15.6a}$$

where ${}^0N^\Omega$ denotes a point whose last n components are solutions of (15.5) (i.e. ${}^0N^r \equiv N^r$) and ${}^0N^0$ is arbitrary, and furthermore (*) denotes any scalar function intrinsic of degree -1 (and $(*)E^\Omega$ is the solution of (15.3)). Without loss of generality we may require

$$*N^\Sigma t_\Sigma = 1 \tag{15.7a}$$

so that

$$*N^\Omega = (*)E^\Omega + {}^0N^\Omega . \tag{15.7b}$$

It follows from (15.7a) that the set of points $*N^\Omega$ constitutes a direction which is not incident with t_Ω. This direction will be referred to as a "projective normal direction." Remembering (10.4) and (13.3b) we see that *for the non-holonomic case this definition is identical with the definition introduced in* §9. From the manner in which the set (15.7b) was obtained, it is clear that the normal direction is of order 3, 4 (2) and any point of (15.7b) is intrinsic of degree -1. Hence

---

[31] One can easily see this using the frame $\underset{t}{U}{}^\Omega$ with $\underset{a}{U}{}^\Omega \equiv U^\Omega_a$, $\underset{\dot n}{U}{}^\Omega \equiv U^\Omega_{\dot n}$, $U^\Omega_{\dot n} t_\Omega = 1$ as we did before on other occasions.

Any point ${}^*N^\Omega$ *satisfying* (15.1) *and* (15.7a) *is contained in the set* (15.7b) *on the normal direction, which is of order* 3, 4, (2). *Furthermore* ${}^*N^\Omega$ *is intrinsic of degree* -1. *In the non-holonomic case, the actual definition of projective normal direction is identical with the definition introduced in* §9, *and consequently this normal direction is preserved point by point by the projectivity, considered in* §§8 *and* 9.

If we denote by ${}^0\Pi^D_{BA}$ the connection induced by the point ${}^0N^\Omega$ of the set (15.7b) then we get from (14.9) and (15.1)

$$(15.8)\qquad {}^0\Pi^e_{BA} = \underset{B}{U^\omega_{,A}} h_{d\omega} h^{de}, \qquad {}^0\Pi^0_{BA} = \frac{1}{C}[\underset{B}{U^0_{,A}} + h_{AB}\,{}^0N^0 - U^0_e \underset{B}{U^\omega_{,A}} h_{d\omega} h^{de}].$$

Hence ${}^0\Pi^e_{BA}$ contains neither ${}^0N^0$ nor $\underset{B}{U^0_{,A}}$ and is of order 3, 4, (2). *Any connection induced by a point of the set* (15.7b) *will be referred to as a "normal connection."* Hence (15.8) is one of the normal connections. If ${}^*\Pi^D_{BA}$ denotes the normal connection induced by ${}^*N^\Omega$ then we have

$$(15.9a)\qquad 1)\ {}^*\Pi^e_{BA} = {}^0\Pi^e_{BA}, \qquad 2)\ {}^*\Pi^0_{BA} = {}^0\Pi^0_{BA} + (*)h_{AB}\frac{E^0}{C}.$$

If we denote by E^A the components of E^Ω in $X^{(n)}_{n-1}$

$$E^\Omega = U^\Omega_A E^A, \qquad E^A = \delta^A_0,$$

then (15.9a) may be condensed into

$$(15.9b)\qquad {}^*\Pi^D_{BA} = {}^0\Pi^D_{BA} + h_{AB}(*)E^D.$$

Hence *the set* (15.7b) *of points induces a set of normal connections* (15.9b). *The components* ${}^*\Pi^e_{BA}$ *of any one of them are of order* 3, 4, (2), *the components* ${}^*\Pi^0_{BA}$ *are of order* 4, 4, (3).

The normal connection ${}^0\Pi^D_{BA}$ induces a point ${}^0X^\Omega$ which is given by

$$(15.10)\qquad {}^0X^\Omega = -\frac{h^{ab}}{n-1}(\underset{b}{U^\Omega_{,a}} - {}^0\Pi^D_{ba}U^\Omega_D)$$

and the normal connection ${}^*\Pi^D_{BA}$ induces the point ${}^*X^\Omega$ which is related to ${}^0X^\Omega$ by

$$(15.11)\qquad {}^*X^\Omega = {}^0X^\Omega + (*)E^\Omega.$$

On the other hand we get from (15.8), (15.10) and (14.8)

$${}^0X^\Omega h_{d\Omega} = -\frac{h^{ab}}{n-1}(\underset{b}{U^\Omega_{,a}} - {}^0\Pi^D_{ba}U^\Omega_D)h_{d\Omega} = -\frac{h^{ab}}{n-1}({}^0\Pi^e_{ba}h_{de} - {}^0\Pi^e_{ba}h_{de}) = 0,$$

$${}^0X^\Omega t_\Omega = 1,$$

and consequently ${}^0X^r = {}^0N^r$. Setting $\Omega = 0$ in (15.10) we have, on account of (15.8)

$$
\begin{aligned}
{}^0X^0 &= -\frac{h^{ab}}{n-1}\,[U^0_{\underset{b}{,a}} - {}^0\Pi^e_{ba}U^0_e - {}^0\Pi^0_{ba}C] \\
&= -\frac{h^{ab}}{n-1}\,[U^0_{\underset{b}{,a}} - U^\lambda_{\underset{b}{,a}}h_{d\lambda}h^{de}U^0_e - U^0_{\underset{b}{,a}} + U^\lambda_{\underset{b}{,a}}h_{d\lambda}h^{de}U^0_e - h_{ab}\,{}^0N^0] = {}^0N^0,
\end{aligned}
$$

and consequently ${}^0X^\Omega = {}^0N^\Omega$. It follows from this and from (15.11) that ${}^*X^\Omega = {}^*N^\Omega$. Hence:

The set (15.9b) *of normal connections induces the set* (15.7) *of normal points*

$$
{}^*N^\Omega = -\frac{h^{ab}}{n-1}\,[U^\Omega_{\underset{b}{,a}} - {}^*\Pi^D_{ba}U^\Omega_D], \tag{15.12a}
$$

where

$$
{}^*N^\lambda \equiv {}^0N^\lambda \equiv -\frac{h^{ab}}{n-1}\,U^\omega_{\underset{b}{,a}}(\delta^\lambda_\omega - h_{d\omega}h^{de}U^\lambda_e). \tag{15.12b}
$$

The last equation gives us the explicit formula for the solution (15.7) *of* (15.1) (*while* (15.12a) *reduces to identity for* $\Omega = 0$).

Let us now consider the normal connection ${}^0\Pi^D_{BA}$. In the same manner as in §14 we may show that there is a point Y^Ω normalized by $Y^\Omega t_\Omega = 1$ and a mixed projective tensor $Z^\Omega_{AB} = Z^D_{AB}U^\Omega_D$ with $Z^\Omega_{AB}t_\Omega = 0$ which satisfy

$$
-h_{AB}Y^\Omega - Z^\Omega_{AB} = U^\Omega_{\underset{B}{,A}} - {}^0\Pi^D_{BA}U^\Omega_D. \tag{15.13}
$$

From (15.12a) and (15.13) we get

$$
Y^\Omega + \frac{h^{ab}}{n-1}\,Z^\Omega_{ab} = {}^0N^\Omega. \tag{15.14}
$$

On the other hand, if we multiply (15.13) by $h_{d\Omega}$ we get

$$
-h_{AB}Y^\Omega h_{d\Omega} - Z^\Omega_{AB}h_{d\Omega} = 0,
$$

and this equation reduces on account of (15.14) and (15.1) to

$$
h_{AB}\frac{h^{ab}}{n-1}\,Z^e_{ab}h_{de} = Z^e_{AB}h_{de},
$$

so that

$$
(n-1)Z^e_{AB} = h_{AB}h^{ab}Z^e_{ab}, \tag{15.15a}
$$

and consequently

$$
Z^\lambda_{AB} = \frac{1}{n-1}\,h_{AB}h^{ab}Z^e_{ab}U^\lambda_e = \frac{1}{n-1}\,h_{AB}h^{ab}Z^\lambda_{ab}. \tag{15.15b}
$$

On the other hand we get from (15.13) for $\Omega = 0$ and from (15.8)

$$-h_{AB}Y^0 - Z^0_{AB} - \underset{B}{U^0_{,A}} + U^0_e \underset{B}{U^\lambda_{,A}} h_{d\lambda} h^{de} = -C\,{}^0\Pi^{\dot{0}}_{BA}.$$

Comparing this equation with the second one of (15.8) we have

$$h_{AB}Y^0 + Z^0_{AB} = h_{AB}\,{}^0N^0$$

and this equation compared with

$$h_{AB}\left(Y^0 + \frac{h^{ab}}{n-1} Z^0_{ab}\right) = h_{AB}\,{}^0N^0,$$

which we get from (15.14) for $\Omega = 0$, gives us

$$Z^0_{AB} = \frac{1}{n-1} h_{AB} h^{ab} Z^0_{ab}. \tag{15.15c}$$

From (15.15b, c) it follows that Z^Ω_{AB} is of the form

$$Z^\Omega_{AB} = Z^\Omega h_{AB} \qquad Z^\Omega t_\Omega = 0, \tag{15.15d}$$

and consequently, in view of (15.14)

$$Y^\Omega + Z^\Omega = {}^0N^\Omega.$$

It follows from this that (15.13) reduces to

$$-h_{AB}\,{}^0N^\Omega = \underset{B}{U^\Omega_{,A}} - {}^0\Pi^D_{BA} U^\Omega_D. \tag{15.16a}$$

Hence ${}^0N^\Omega$ and Π^D_{BA} are in involution. From (15.9b), (15.7b) and (15.16a) we get

$$-h_{AB}\,{}^*N^\Omega = \underset{B}{U^\Omega_{,A}} - {}^*\Pi^D_{BA} U^\Omega_D. \tag{15.16b}$$

Hence:

The normal point (15.12a) *and the normal connection* ${}^*\Pi^D_{BA}$ *induced by it are in involution.*

NOTE. From (15.16) we get

$${}^*\Pi^D_{0A} = {}^*\Pi^D_{A\dot{0}} = {}^0\Pi^D_{\dot{0}A} = {}^0\Pi^D_{A\dot{0}} = Cc\delta^D_A. \tag{15.17}$$

16. Privileged normal connection

In order to choose conveniently the arbitrary function (*) in (15.9b) we may proceed in two different ways. We may first choose conveniently one of the points (15.7) and get in this way a fixed connection. Or we may prescribe some condition on the connection ${}^*\Pi^D_{AB}$ itself which gives us the function (*). Let us initially apply the first method. In §13 we defined a privileged quadratic

tensor, of order 4, 5, (3). Let us denote it here by $G_{\Omega\Sigma}$ and let us choose such a point ${}^*N^\Omega$ in the set (15.7) that it will be incident with the corresponding quadric

$$ {}^*N^\Omega\, {}^*N^\Sigma G_{\Omega\Sigma} = 0.^{32} $$

This equation gives us

$$ {}^*N^0 = -\frac{1}{2c}\, {}^*N^\lambda\, {}^*N^\mu G_{\lambda\mu} $$

or, according to (15.12b)

$$ {}^*N^0 = -\frac{h^{ab}h^{ed}}{2c(n-1)^2} G_{\lambda\mu} U^\alpha_{b,a} U^\beta_{d,e} (\delta^\lambda_\alpha - h_{c\alpha}h^{cf}U^\lambda_f)(\delta^\mu_\beta - h_{i\beta}h^{ij}U^\mu_j). \tag{16.1} $$

Substituting these values in the expression for ${}^*\Pi^{\dot 0}_{BA}$ (analogous to the second of (15.8)) we get

$$ \begin{aligned} {}^*\Pi^{\dot 0}_{BA} = \frac{1}{C}\Bigg[& U^0_{B,A} - U^0_e U^\omega_{B,A} h_{d\omega}h^{de} \\ & - h_{AB}\frac{h^{ab}h^{ed}}{2C(n-1)^2} G_{\lambda\mu} U^\alpha_{b,a} U^\beta_{d,e} (\delta^\lambda_\alpha - h_{c\alpha}h^{cf}U^\lambda_f)(\delta^\mu_\beta - h_{i\beta}h^{ij}U^\mu_j)\Bigg] \end{aligned} \tag{16.2} $$

while ${}^*\Pi^e_{BA} = {}^0\Pi^e_{BA}$. The corresponding induced point in involution with ${}^*\Pi^D_{BA}$ is defined by (15.12b) and (16.1). From the manner in which ${}^*\Pi^{\dot 0}_{BA}$ and ${}^*N^0$ were obtained, it is clear that both objects are of order 4, 5, (3). Hence:

In the set (15.9b) *of normal connections, there is a privileged one, namely the one induced and in involution with the point* ${}^*N^\Omega$ *of* (15.7) *which lies on the quadric defined by means of the privileged projective tensor of* §13. *This connection is given by the first set of* (15.9a) *and by* (16.2). *Its coefficients* ${}^*\Pi^e_{BA}$ *are of order* 3, 4, (2) *while* ${}^*\Pi^{\dot 0}_{BA}$ *are of order* 4, 5, (3). *The corresponding normal point is given by* (15.12b) *and* (16.1). ${}^*N^\lambda$ *are of order* 3, 4, (2) *while* ${}^*N^0$ *is of order* 4, 5, (3).

Let us now apply the second method which consists in prescribing a convenient condition on the desired connection. In order to do that we confine ourselves to the *holonomic* case of an X_{n-1}. Then we may introduce the curvature projective tensor ${}^0P^A_{EFB}$ of ${}^0\Pi^D_{BA}$

$$ {}^0P^A_{EFB} \equiv \partial_F\, {}^0\Pi^A_{BE} - \partial_E\, {}^0\Pi^A_{BF} + {}^0\Pi^A_{DF}\, {}^0\Pi^D_{BE} - {}^0\Pi^A_{DE}\, {}^0\Pi^D_{BF} \tag{16.3a} $$

and the contracted projective tensor

$$ {}^0P_{AB} = P^E_{EAB} \tag{16.3b} $$

which, in consequence of (15.17) satisfies the following equation

$$ {}^0P_{AB} = \delta^a_A \delta^b_B\, {}^0P_{ab}. \tag{16.3c} $$

---

[32] We confine ourselves to non-trivial solutions of this equation and exclude also the point E^Ω taking for ${}^*N^\lambda$ the values (15.12b).

If we denote by ${}^*P_{AB}$ the contracted projective curvature tensor, defined by an equation analogous to (16.3b), this tensor obeys the equations analogous to (16.3c) and moreover we have, according to (15.9b)

$$ {}^*P_{AB} = {}^0P_{AB} - Cc(n-2)(*)h_{AB}. \tag{16.4} $$

Now we choose the arbitrary function (*) in such a way that

$$ {}^*P_{AB}h^{AB} = 0. \tag{16.5} $$

The necessary and sufficient condition that this equation be satisfied is

$$ (*) = \frac{{}^0P_{ab}h^{ab}}{Cc(n-1)(n-2)} $$

and this equation may be written according to (6.8) in the following form

$$ (*) = \frac{{}^0P_{ab}h^{ab}}{c'n(n-2)}. \tag{16.6} $$

Substituting this value in (15.9b) we get the corresponding ${}^*\Pi^D_{BA}$ which we denote now simply by Π^D_{BA}

$$ \Pi^D_{BA} = {}^0\Pi^D_{BA} + h_{AB}E^D \frac{{}^0P_{ab}h^{ab}}{c'n(n-2)}. \tag{16.7} $$

The contracted projective curvature tensor of Π^D_{BA} will be denoted also simply by P_{AB}. It satisfies the equation

$$ P_{AB}h^{AB} = P_{ab}h^{ab} = 0 \tag{16.8} $$

which is equivalent to (16.5). It can be easily shown that this connection, defined by (16.7) does not depend on the particular choice of ${}^0\Pi^D_{BA}$. Moreover from the manner in which it was obtained it is clear that its coefficients Π^0_{BA} are of order 4, 5. Denoting by N^Ω the point in involution with (16.7) we get from (15.7) and (16.6)

$$ N^\Omega = \frac{{}^0P_{ab}h^{ab}}{c'n(n-2)} E^\Omega + {}^0N^\Omega. \tag{16.9} $$

It can easily be shown that this result does not depend on the particular choice of ${}^0N^0$ (the remaining components ${}^0N^r$ of ${}^0N^\Omega$ being given by (15.12b)). Obviously N^0 is of order 4, 5. Hence we have the following theorem in the holonomic case: *In the set* (15.9b) *of normal connections there is one privileged in the sense that its contracted projective curvature tensor* P_{AB} *satisfies* (16.8). *This connection* Π^D_{BA} *is given by* (16.7) *and its coefficients* Π^e_{BA} *and* Π^0_{BA} *are of order* 3, 4 *and* 4, 5. *It is in involution with the point* (16.9), *whose components* N^λ *are of order* 3, 4 *while* N^0 *is of order* 4, 5.

17. The absolute connection

The equation (16.4) enables us to eliminate (*) from (15.9b). For we have

$$(*)h_{AB} = \frac{{}^*P_{AB} - {}^0P_{AB}}{Cc(2-n)}. \tag{17.1}$$

Substituting this value in (15.9b) we find that the connection

$$\Omega^D_{BA} = {}^0\Pi^D_{BA} + \frac{{}^0P_{AB}}{Cc(n-2)}E^D = {}^*\Pi^D_{BA} + \frac{{}^*P_{AB}}{Cc(n-2)}E^D \tag{17.2a}$$

is independent of the particular choice of (*), i.e., is independent of the particular choice of the normal connection by means of which it is defined. This connection will be referred to as an "absolute connection." From (17.2a) and (16.3c) we get

$$\Omega^D_{\overset{0}{b}A} = \Omega^D_{A\overset{0}{b}} = Cc\delta^D_A. \tag{17.2b}$$

The absolute connection induces a point

$$A^\Sigma = -\frac{h^{ab}}{n-1}\,(\underset{b}{U}{}^\Sigma_{,a} - \Omega^D_{ba}U^\Sigma_D). \tag{17.3}$$

Using (17.2a), we may write for (17.3) according to (16.9) and (6.8)

$$A^\Omega = {}^0N^\Omega + \frac{h^{ab}\,{}^0P_{ab}}{(n-1)Cc(n-2)}E^D U^\Omega_D = {}^0N^\Omega + \frac{{}^0P_{ab}h^{ab}}{c'n(n-2)}E^\Omega = N^\Omega. \tag{17.4}$$

Consequently the absolute connection induces the same normal point as the privileged normal connection, *but is not in involution with it.* Indeed, using the fact that Π^D_{BA} and N^Ω are in involution

$$-h_{AB}N^\Omega = \underset{B}{U}{}^\Omega_{,A} - \Pi^D_{BA}U^\Omega_D$$

we may write, making use of (16.7) and (17.2a)

$$\begin{aligned} h_{AB}N^\Omega + \underset{B}{U}{}^\Omega_{,A} &= {}^0\Pi^D_{BA}U^\Omega_D + h_{AB}E^\Omega\frac{{}^0P_{ab}h^{ab}}{c'n(n-2)} \\ &= \Omega^D_{BA}U^\Omega_D - \frac{{}^0P_{AB}}{Cc(n-2)}E^\Omega + h_{AB}E^\Omega\frac{{}^0P_{ab}h^{ab}}{c'n(n-2)}, \end{aligned}$$

and this equation is equivalent to

$$-h_{AB}N^\Omega - \frac{{}^0P_{ab}}{Cc(n-2)}\left[\delta^a_A\delta^b_B - h_{AB}\frac{h^{ab}}{n-1}\right]E^\Omega = \underset{B}{U}{}^\Omega_{,A} - \Omega^D_{BA}U^\Omega_D. \tag{17.5}$$

This equation shows us that N^Ω and Ω^D_{BA} are not in involution. On the other hand we get from (16.4)

$${}^*P_{ab}\left(\delta^a_A\delta^b_B - h_{AB}\frac{h^{ab}}{n-1}\right) = {}^0P_{ab}\left(\delta^a_A\delta^b_B - h_{AB}\frac{h^{ab}}{n-1}\right),$$

so that (17.5) does not depend on the particular choice of the normal connection whose contracted projective curvature tensor appears in it. Hence, recapitulating, we have the following theorem:

The absolute connection Ω^D_{BA} given by (17.2a) *is independent of the choice of the normal connection which appears in its definition. Its coefficients Ω^e_{BA} and $\Omega^{\dot 0}_{BA}$ are of order* 3, 4 *and* 4, 5. *It induces the same point as the privileged normal connection* (16.7) *but is not in involution with it, as equation* (17.5) *shows, this equation being independent of the choice of the normal connection whose contracted curvature tensor appears in it.*

18. Weyl's projective connection

The connections considered up to this point were constructed by means of $U^{\Omega}_{B,A}$. But we may construct a projective connection without this object. In order to do that let us again confine ourselves to the *holonomic* case and let us denote by $\Gamma^D_{BA} = \Gamma^D_{AB}$ the components of the desired connection. If k is any real number, and g_{ABD} denotes the projective tensor defined by (11.5b), which obeys the law (12.1), and Q_A is defined by (6.13b), then obviously

$$\partial_A h_{BD} - \Gamma^E_{BA} h_{ED} - \Gamma^E_{DA} h_{BE} + Q_A h_{BD} - k g_{ABD} = 0 \tag{18.1}$$

is an intrinsic and covariant equation. Proceeding in the usual manner we get from (18.1)

$$\Gamma^e_{BA} = \tfrac{1}{2}\delta^a_A \delta^b_B \left\{ {e \atop ba} \right\} + \tfrac{1}{2}(Q_A \delta^e_B + Q_B \delta^e_A - Q_D h^{De} h_{AB}) - \frac{k}{2} \delta^a_A \delta^b_B (g_{abd} + g_{bda} - g_{dab}) h^{de}, \tag{18.2a}$$

where

$$\left\{ {e \atop ba} \right\} = \tfrac{1}{2} h^{de} (\partial_a h_{bd} + \partial_b h_{da} - \partial_d h_{ab}).$$

From (18.2a) we get

$$\Gamma^e_{0A} = \Gamma^e_{A\dot 0} = \tfrac{1}{2} Q_{\dot 0} \delta^e_A = C c \delta^e_A .$$

It follows from the transformation law for Γ^D_{BA} that Γ^D_{0A} $(= \Gamma^D_{A\dot 0})$ is a projective tensor. Therefore we may complete the foregoing equation, writing

$$\Gamma^D_{0A} = \Gamma^D_{A\dot 0} = C c \delta^D_A . \tag{18.2b}$$

In order to define the remaining coefficients $\Gamma^{\dot 0}_{ba}$ let us first introduce the curvature tensor G^A_{EFB} of Γ^D_{BA}

$$G^A_{EFB} \equiv \partial_F \Gamma^A_{BE} - \partial_E \Gamma^A_{BF} + \Gamma^A_{DF} \Gamma^D_{BE} - \Gamma^A_{DE} \Gamma^D_{BF}$$

[33] According to $h_{AB} = \delta^a_A \delta^b_B h_{ab}$ the coefficients Γ^0_{AB} do not appear in (18.1).

and its contracted tensor

$$G_{FB} \equiv G^{A}_{AFB}, \tag{18.3a}$$

which satisfies the following equations

$$G_{FB} = \delta^{f}_{F}\delta^{b}_{B}G_{fb}, \tag{18.3b}$$

where G_{fb} is given by

$$\begin{aligned} G_{fb} &= \partial_f\Gamma^{a}_{ba} - \partial_a\Gamma^{a}_{bf} + \Gamma^{A}_{Df}\Gamma^{D}_{bA} - \Gamma^{A}_{DA}\Gamma^{D}_{bf} \\ &= \partial_f\Gamma^{a}_{ba} - \partial_a\Gamma^{a}_{bf} + \Gamma^{a}_{df}\Gamma^{d}_{ba} - \Gamma^{a}_{da}\Gamma^{d}_{bf} + Cc(2-n)\dot{\Gamma}^{0}_{bf}. \end{aligned} \tag{18.3c}$$

In this formula the only unknown expressions on the right hand side are $\dot{\Gamma}^{0}_{bf}$. We may choose them in such a way that $G_{bf} = 0$. From (18.3b) it follows that this requirement is independent of the choice of the y's. The foregoing equation is satisfied if we choose $\dot{\Gamma}^{0}_{ba}$ according to

$$\dot{\Gamma}^{0}_{ba} = \frac{1}{Cc(n-2)}\left(\partial_a\Gamma^{d}_{bd} - \partial_d\Gamma^{d}_{ba} + \Gamma^{e}_{da}\Gamma^{d}_{bc} - \Gamma^{e}_{dc}\Gamma^{d}_{ba}\right) \tag{18.2c}$$

and this equation is independent of the choice of the y's. Because Γ^{d}_{ba} is of order 3, 4, $\dot{\Gamma}^{0}_{ba}$ is of order 4, 5. If we denote by $\phi|_A$ the covariant derivative of ϕ with respect to Γ^{D}_{BA}, then (18.1) may be written

$$h_{BD|A} = -(Q_A h_{BD} - k g_{ABD}). \tag{18.4}$$

We see from (18.4) that for $k = 0$ we get a projective connection which is analogous to Weyl's affine connection. Therefore we refer to Γ^{D}_{BA} as "Weyl's projective connection."

If k is any real number, then the requirement (18.4) *and* $G_{FB} = 0$ *enables us to construct a set of projective Weyl's connections (one for any k). Its coefficients are given by* (18.2). Γ^{d}_{ba} *are of order* 3, 4 *while* $\dot{\Gamma}^{0}_{ba}$ *are of order* 4, 5.

PRINCETON UNIVERSITY.

HYPERSURFACES OF A SPACE OF CONSTANT CURVATURE[1]

By Aaron Fialkow

(Received February 14, 1938)

1. Introduction

In this paper we study the geometric properties of those real Riemann spaces which may be imbedded as hypersurfaces of a real space of constant curvature. In particular, the results obtained apply to hypersurfaces of a flat space, i.e., to spaces of class one. Several of the geometric properties which are derived are individually characteristic of spaces of constant curvature in the sense that if any one of these properties is enjoyed by all hypersurfaces of a Riemann space, the enveloping space has constant curvature.

The elements of the theory of hypersurfaces which are necessary for our work are stated in §2. It is assumed that the functions defining the imbedding have enough continuous derivatives to insure the validity of the Gauss and Codazzi equations. Furthermore, we assume that all the principal normal curvatures of the hypersurface are real and that none of the lines of curvature is tangent to a null vector. Of course, if the first fundamental form of the enveloping space is definite, this last assumption is always satisfied and is no restriction.

It is known that if a hypersurface[2] V_N is a space of constant curvature S_{N+1} admits N real orthogonal congruences of lines of curvature, their tangents are Ricci principal directions for V_N. For $N > 2$, we prove in §3 that, conversely, only if the enveloping space is an S_{N+1} do all its hypersurfaces have this property.

In §4, some simple properties of a subspace V_P of a V_N immersed in a V_M are obtained which are of use in later sections. In §5, some general properties of the lines of curvature of the hypersurfaces V_N of an S_{N+1} are derived. Thus we show that in general if P $(1 < P < N)$ congruences of lines of curvature of V_N lie in $\infty^{N-P} V_P$, these congruences are also lines of curvature of the V_P. A similar relation holds for the Ricci principal directions of V_N. If several of the principal normal curvatures are equal, the corresponding properties of V_N are found in §6.

In the following section, we give a complete classification of the Einstein spaces E_N in an S_{N+1} subject to the restrictions of §2. This problem is completely solved since both the first fundamental forms and the equations of imbedding of all such spaces are found. Finally in §8, the algebraic characteriza-

---

[1] Most of the results contained in §§3, 6 and 7 of this paper were obtained while the author was a National Research Fellow at Princeton University and the Institute for Advanced Study.

[2] We denote an N-dimensional Riemann space, Einstein space and space of constant curvature by V_N, E_N and S_N respectively.

tion of the E_N's in an S_{N+1} is discussed. References to other writers on these subjects are given below.

2. Hypersurfaces of a V_{N+1}

We first give a short summary of those parts of the theory of hypersurfaces which are necessary for what follows. Let V_{N+1} be a real Riemann space whose first fundamental form is[3]

$$ds^2 = f_{\alpha\beta}dy^\alpha dy^\beta. \tag{2.1}$$

Any real hypersurface V_N immersed in V_{N+1} is defined by a system of equations

$$y^\alpha = y^\alpha(x^1, x^2, \cdots, x^N),$$

where the rank of the functional matrix $\| \partial y^\alpha/\partial x^i \|$ is N. The first fundamental form of V_N is

$$ds^2 = g_{ij}dx^i dx^j \tag{2.2}$$

where[4]

$$g_{ij} = f_{\alpha\beta}y^\alpha_{,i}y^\beta_{,j}. \tag{2.3}$$

We assume that coördinate systems $\{y^\alpha\}$ and $\{x^i\}$ and an open simply connected region $\mathfrak{R}$ of the N-dimensional arithmetic number space $\{x^i\}$ exists such that $f_{\alpha\beta}(y^\gamma)$ are real functions of class C^2 and $y^\alpha(x^i)$ are real functions of class C^3 for $x \subset \mathfrak{R}$. Since the rank of $\| \partial y^\alpha/\partial x^i \|$ is N, (2.2) is non-singular for $x \subset \mathfrak{R}$ although it may be indefinite. Of course, instead of the particular coördinate systems indicated above any other 3-systems[5] may be used; i.e., coördinate transformations must be real and possess continuous partial derivatives up to the third order inclusive. In the remainder of the paper it is to be understood that the coördinates y^α; x^i refer to points of the region $\mathfrak{R}$. The properties of V_N which we derive are valid throughout $\mathfrak{R}$.

Under these conditions the Gauss equations

$$R_{ijkl} = e(\Omega_{ik}\Omega_{jl} - \Omega_{il}\Omega_{jk}) + \bar{R}_{\alpha\beta\gamma\delta}y^\alpha_{,i}y^\beta_{,j}y^\gamma_{,k}y^\delta_{,l} \tag{2.4}$$

---

[3] Unless it is stated otherwise, throughout this paper the indices $\alpha, \beta, \gamma, \delta; i, j, k, l, m, n; p, q, r, s; a, b, c, d; \sigma$ have the ranges $1, 2, \cdots, M; 1, 2, \cdots, N; 1, 2, \cdots, P; P+1, P+2, \cdots, N$ and $N+1, N+2, \cdots; M$ respectively. In all sections except §4, $M = N+1$. It is to be understood that a tensor equation in which an index is not summed is valid for each value of the index within its range. A covariant or contravariant index which appears twice in an expression is to be summed over the appropriate range.

[4] The comma denotes covariant differentiation with respect to the y's and the form (2.1) or the x's and (2.2) or the w's and (4.1) according as the range of the covariant index following the comma is $1, 2, \cdots, M$ or $1, 2, \cdots, N$ or $1, 2, \cdots, P$ respectively.

[5] For the definitions and properties of an m-system, cf. Duschek-Mayer, *Lehrbuch der Differentialgeometrie*, 1930, vol. 2, pp. 1–2.

are satisfied. In these equations, R_{ijkl} and $\bar{R}_{\alpha\beta\gamma\delta}$ are the Riemann curvature tensors of V_N and V_{N+1} respectively, Ω_{ij} are the coefficients of the second fundamental form of V_N and e is $+1$ or -1 being defined by

$$e = f_{\alpha\beta}\,\xi^\alpha\,\xi^\beta \tag{2.5}$$

where ξ^α are the components of the unit normal to V_N. The functions R_{ijkl} and $\bar{R}_{\alpha\beta\gamma\delta}$ are of class C^0, Ω_{ij} of class C^1 and ξ^α of class C^2.

The directions of the lines of curvature of V_N are given by the vectors ${}_{(m)}\lambda^{i}$[6] which satisfy

$$(\Omega_{ij} - \rho_m g_{ij})_{(m)}\lambda^i = 0, \tag{2.6}$$

where ρ_m is a principal normal curvature and is a root of the determinant equation

$$| \Omega_{ij} - \rho g_{ij} | = 0. \tag{2.7}$$

The ρ_m need not be real if the form (2.2) is indefinite. If the elementary divisors of (2.7) are simple (as is always the case if (2.2) is definite), there is at least one orthogonal ennuple of unit vectors $(m)\lambda^i$ which satisfy (2.6). When one or more of the elementary divisors are not simple, the tangents to some of the lines of curvature are null vectors.[7] If the principal normal curvatures are real and none of the lines of curvature are tangent to null vectors, we call V_N a *proper* hypersurface of V_{N+1} and term the imbedding a *proper* imbedding. Similarly we use the word *improper* to denote not proper in the above sense. If the first fundamental form of the enveloping V_{N+1} is definite, it follows that every hypersurface is proper.

3. A characteristic property of an S_{N+1}

It was shown by Schouten and Struik[8] that the lines of curvature of a V_N in a Euclidean S_{N+1} are also Ricci principal directions of V_N. Eisenhart[9] showed that this theorem is true for a proper V_N in any S_{N+1}. In this section, we prove the converse of this theorem. The direct and converse results together constitute a characterization of spaces of constant curvature by means of their hypersurfaces. The complete theorem is stated in

---

[6] Here m denotes the vector and i the component.

[7] For the algebraic details of this theory, cf. T. J. I'A. Bromwich, *Quadratic forms and their classification by means of invariant factors*, (1906), chapters 3 and 4; M. Bôcher, *Introduction to Higher Algebra*, (1929), p. 305. For the geometric aspects, cf. L. P. Eisenhart, *Riemannian Geometry*, (1926), pp. 107–113.

[8] J. A. Schouten and D. J. Struik, *On some properties of general manifolds relating to Einstein's theory of gravitation*, American Journal of Mathematics, vol. 43 (1921), p. 214.

[9] Eisenhart, *loc. cit.*, p. 199 and p. 219 ex. 7.

THEOREM 3.1. *The tangents to the lines of curvature of a proper hypersurface V_N of an S_{N+1} are Ricci principal directions for the V_N*[10]. *Conversely, if every proper hypersurface of a V_{N+1} $(N > 2)$ has this property, the V_{N+1} is an S_{N+1}.*

The converse statement in this theorem contradicts an erroneous theorem due to Struik[11] which states that the hypersurfaces of any Einstein space whose first fundamental form is positive definite have the above property. For $N = 2$, every direction of the V_N is a Ricci principal direction so that the directions of the lines of curvature of a V_2 in any V_3 are always tangent to Ricci principal directions of V_2. This explains the hypothesis $N > 2$. The proof of the theorem follows.

The Ricci tensor R_{ij} is defined as $g^{il}R_{ijkl}$. It follows from (2.4) that

$$R_{jk} = eg^{il}(\Omega_{ik}\Omega_{jl} - \Omega_{il}\Omega_{jk}) + \bar{R}_{\alpha\beta\gamma\delta}y^{\alpha}_{,i}y^{\beta}_{,j}y^{\gamma}_{,k}y^{\delta}_{,l}g^{il}. \tag{3.1}$$

Since V_N is proper, at any point A there is at least one set of N mutually orthogonal unit vectors ${}_{(m)}\lambda^i$ which are tangent to lines of curvature at A. These directions may be chosen as coördinate directions by a suitable real transformation. We denote the corresponding coördinate system $\{x^i\}$ by X. It follows from (2.6) that, at A, in the coördinate system X,

$$\begin{aligned} &g_{ij} = l_j; \quad g_{jk} = 0; \quad {}_{(j)}\lambda^j = 1; \\ &{}_{(k)}\lambda^j = 0; \quad \Omega_{jj} = e_j\rho_j \quad \Omega_{jk} = 0, \end{aligned} \qquad (j \neq k), \tag{3.2}$$

where

$$e_m = g_{ij}\,{}_{(m)}\lambda^i\,{}_{(m)}\lambda^j. \tag{3.3}$$

If these results are substituted in (3.1), we have

$$\begin{aligned} R_{jj} &= ee_j\rho_j \sum_{k\neq j} \rho_k + \bar{R}_{\alpha\beta\gamma\delta}y^{\alpha}_{,i}y^{\beta}_{,j}y^{\gamma}_{,j}y^{\delta}_{,l}g^{il}, \\ R_{jk} &= \bar{R}_{\alpha\beta\gamma\delta}y^{\alpha}_{,i}y^{\beta}_{,j}y^{\gamma}_{,k}y^{\delta}_{,l}g^{il}, \qquad (j \neq k). \end{aligned} \tag{3.4}$$

From (3.2), it follows that the tangents at A to the lines of curvature of V_N will be Ricci principal directions when and only when $R_{jk} = 0 (j \neq k)$. This characteristic equation, in consequence of the second equations of (3.4), is

$$\bar{R}_{\alpha\beta\gamma\delta}y^{\alpha}_{,i}y^{\beta}_{,j}y^{\gamma}_{,k}y^{\delta}_{,l}g^{il} = 0, \qquad (j \neq k). \tag{3.5}$$

It is easy to verify that (3.5) is true identically if V_{N+1} is an S_{N+1} which proves the theorem of Schouten and Struik.

---

[10] More generally, it can be shown that if the tangent of a real line of curvature of a V_N (not necessarily proper) in an S_{N+1} is not a null vector, it is a Ricci principal direction of V_N.

[11] D. J. Struik, *Grundzüge der mehrdimensionalen Differentialgeometrie in direkter Darstellung*, (1922), p. 166. On the same page the result on E_N's in an E_{N+1} for which Ω_{ij} has rank 0 or 1 is incorrect. Also, the second equation in Schouten and Struik, *loc. cit.*, p. 215 and the deductions made from it are fallacious. In all cases, the error which is made is equivalent to omitting the term $e\xi^{\alpha}\xi^{\beta}$ in (3.9).

To prove the converse theorem, we proceed to simplify the left hand member of (3.5). It is well known[12] that the metric tensor has the canonical representation

$$g^{il} = \sum_{m=1}^{N} e_m \, {}_{(m)}\lambda^i \, {}_{(m)}\lambda^l. \tag{3.6}$$

If we let ${}_{(m)}\xi^\alpha$ denote the components in the y's of the ${}_{(m)}\lambda^i$

$${}_{(m)}\xi^\alpha = {}_{(m)}\lambda^j y^\alpha_{,j}. \tag{3.7}$$

It follows from (2.3), (3.3) and (3.7) that

$$e_m = f_{\alpha\beta} \, {}_{(m)}\xi^\alpha \, {}_{(m)}\xi^\beta.$$

Therefore

$$f^{\alpha\delta} = \sum_{m=1}^{N} e_m \, {}_{(m)}\xi^\alpha \, {}_{(m)}\xi^\delta + e\xi^\alpha\xi^\delta. \tag{3.8}$$

From (3.6), (3.7) and (3.8) we have

$$y^\alpha_{,i} y^\delta_{,l} g^{il} = f^{\alpha\delta} - e\xi^\alpha\xi^\delta. \tag{3.9}$$

In the coördinate system X, it follows from (3.7) that

$${}_{(m)}\xi^\alpha = y^\alpha_{,m}. \tag{3.10}$$

In consequence of (3.9) and (3.10), the condition (3.5) may be written as

$$\bar{R}_{\alpha\beta\gamma\delta} \, {}_{(j)}\xi^\beta \, {}_{(k)}\xi^\gamma (f^{\alpha\delta} - e\xi^\alpha\xi^\delta) = 0, \qquad (j \neq k). \tag{3.11}$$

It is easy to show that a proper V_N exists the direction of whose lines of curvature at A is any given orthogonal ennuple.[13] Since (3.11) must hold for every proper V_N, we may think of ${}_{(j)}\xi^\alpha$, ${}_{(k)}\xi^\alpha$ and ξ^α as any arbitrary orthogonal triplet of vectors at A. From these remarks it follows that ξ^α may be replaced by ${}_{(m)}\xi^\alpha$ where m is not j or k. This gives

$$\bar{R}_{\alpha\beta\gamma\delta} \, {}_{(j)}\xi^\beta \, {}_{(k)}\xi^\gamma (f^{\alpha\delta} - e_{m(m)} \, {}_{(m)}\xi^\alpha \, {}_{(m)}\xi^\delta) = 0, \qquad (j, k, m \neq).$$

In this equation, let m range over the $(N - 2)$ indices different from j and k and add the resulting equations to (3.11). We obtain

$$\bar{R}_{\alpha\beta\gamma\delta} \, {}_{(j)}\xi^\beta \, {}_{(k)}\xi^\gamma \Big[(N-1) f^{\alpha\delta} - \sum_{m\neq j,k} e_m \, {}_{(m)}\xi^\alpha \, {}_{(m)}\xi^\delta - e\xi^\alpha\xi^\delta \Big] = 0. \tag{3.12}$$

Now

$$\bar{R}_{\alpha\beta\gamma\delta} \, {}_{(j)}\xi^\beta \, {}_{(k)}\xi^\gamma [e_j \, {}_{(j)}\xi^\alpha \, {}_{(j)}\xi^\delta + e_k \, {}_{(k)}\xi^\alpha \, {}_{(k)}\xi^\delta] = 0.$$

---

[12] Duschek-Mayer, *loc. cit.*, p. 26 and Eisenhart, *loc. cit.*, p. 96.

[13] A simple proof of this statement appears in a paper by the author entitled *The Riemannian curvature of a hypersurface*, Bulletin of the American Mathematical Society, April 1938, pp. 253–257.

In consequence of this equation and (3.8), (3.12) is equivalent to

$$(N-2)\,\bar{R}_{\alpha\beta\gamma\delta\,(j)}\xi^{\beta}{}_{(k)}\xi^{\gamma}f^{\alpha\delta}=0.$$

Since $N>2$, (3.11) becomes

$$\bar{R}_{\alpha\beta\gamma\delta}\xi^{\alpha}{}_{(j)}\xi^{\beta}{}_{(k)}\xi^{\gamma}\xi^{\delta}=0, \tag{3.13}$$

where ξ^{α}, ${}_{(j)}\xi^{\alpha}$ and ${}_{(k)}\xi^{\alpha}$ are arbitrary mutually orthogonal vectors. It remains to show that the only V_{N+1}'s obeying (3.13) are spaces of constant curvature. After a change of notation, this objective is stated in

THEOREM 3.2. *A necessary and sufficient condition that*

$$R_{ijkl\,(1)}\lambda^{i}{}_{(2)}\lambda^{j}{}_{(1)}\lambda^{k}{}_{(3)}\lambda^{l}=0 \tag{3.14}$$

for every orthogonal triplet of vectors ${}_{(t)}\lambda^{i}$ $(t=1,2,3)$ *in a* V_N $(N>2)$ *is that* V_N *be an* S_N.

This result is closely analogous to a theorem due to Schouten[14] which characterizes conformally flat spaces. The proof follows. By hypothesis

$$\begin{aligned} &g_{ij\,(t)}\lambda^{i}{}_{(u)}\lambda^{j}=0, \qquad (t\neq u),\\ &g_{ij\,(t)}\lambda^{i}{}_{(t)}\lambda^{j}=e_t, \end{aligned} \qquad (t,u=1,2,3). \tag{3.15}$$

If V_N is a space of constant curvature,

$$R_{ijkl}=K(g_{ik}g_{jl}-g_{il}g_{jk})$$

and (3.14) is a consequence of (3.15).

Conversely, let (3.14) be satisfied for every orthogonal triplet of vectors ${}_{(t)}\lambda^{i}$. Since ${}_{(1)}\lambda^{i}$, $(e_{2\,(2)}\lambda^{i}+e_{3\,(3)}\lambda^{i})$, $(-{}_{(2)}\lambda^{i}+{}_{(3)}\lambda^{i})$ is such an orthogonal triplet,

$$R_{ijkl\,(1)}\lambda^{i}(e_{2\,(2)}\lambda^{j}+e_{3\,(3)}\lambda^{j})_{(1)}\lambda^{k}(-{}_{(2)}\lambda^{l}+{}_{(3)}\lambda^{l})=0. \tag{3.16}$$

Since the Riemannian curvature of V_N determined by two vectors μ^{i}, ν^{i} is

$$[\mu^{i},\nu^{i}]=\frac{R_{ijkl}\mu^{i}\nu^{j}\mu^{k}\nu^{l}}{(g_{ik}g_{jl}-g_{il}g_{jk})\mu^{i}\nu^{j}\mu^{k}\nu^{l}}, \tag{3.17}$$

it follows from (3.14), (3.15) and (3.16) that

$$[{}_{(1)}\lambda^{i},{}_{(2)}\lambda^{i}]=[{}_{(1)}\lambda^{i},{}_{(3)}\lambda^{i}]=K_1. \tag{3.18}$$

If ζ^{i} is any vector orthogonal to ${}_{(1)}\lambda^{i}$, two vectors ${}_{(2)}\lambda^{i}$, ${}_{(3)}\lambda^{i}$ exist which satisfy (3.15) such that

$$\zeta^{i}=a_{(2)}\lambda^{i}+b_{(3)}\lambda^{i}.$$

From (3.14), (3.17) and (3.18),

$$[{}_{(1)}\lambda^{i},\zeta^{i}]=K_1. \tag{3.19}$$

---

[14] J. A. Schouten, *Über die konforme Abbildung n-dimensionaler Mannigfaltigkeiten mit quadratischer Massbestimmung auf eine Mannigfaltigkeit mit euklidischer Massbestimmung*, Mathematische Zeitschrift, vol. 11 (1921), p. 84.

Since the assumption that ζ^i is orthogonal to ${}_{(1)}\lambda^i$ is no real restriction, the Riemannian curvature is the same for all orientations which contain any given vector ${}_{(1)}\lambda^i$. By repeated use of (3.19) we have

$$[\lambda^i, \zeta^i] = [\lambda^i, \mu^i] = [\mu^i, \lambda^i] = [\mu^i, \nu^i],$$

where the four vectors are arbitrary. Hence V_N has constant curvature. This completes the proofs of Theorem 3.1 and Theorem 3.2.

Theorem 3.1 shows that the ${}_{(m)}\lambda^i$ are Ricci principal directions as well as lines of curvature of V_N in the S_{N+1}. Hence the following canonical representations are valid:

$$g_{ij} = \sum_{m=1}^{N} e_m {}_{(m)}\lambda_{i\,(m)}\lambda_j,$$

$$\Omega_{ij} = \sum_{m=1}^{N} e_m \rho_m {}_{(m)}\lambda_{i\,(m)}\lambda_j,$$

$$R_{ij} = \sum_{m=1}^{N} e_m \kappa_m {}_{(m)}\lambda_{i\,(m)}\lambda_j,$$

where the ρ_m and κ_m are principal normal and principal Ricci curvatures respectively. Since

$$\kappa_m = -\frac{R_{ij\,(m)}\lambda^{i}{}_{(m)}\lambda^j}{g_{ij\,(m)}\lambda^{i}{}_{(m)}\lambda^j},$$

and

$$\bar{R}_{\alpha\beta\gamma\delta} = K_0(f_{\alpha\gamma}f_{\beta\delta} - f_{\alpha\delta}f_{\beta\gamma}),$$

we find from (3.4) that

$$\kappa_m = e\rho_m \sum_{k\neq m} \rho_k + (N-1)K_0. \tag{3.20}$$

It is easy to show that if P and only P of the κ's are equal at a point A, then P congruences of lines of curvature at A lie in the linear vector space determined by the Ricci principal directions corresponding to the equal κ's. In particular, if κ_1 does not equal any of the other κ's, the corresponding congruence of Ricci principal directions is also a congruence of lines of curvature.

4. A subspace V_P of a V_N in a V_M

Some simple properties of a subspace V_P of a V_N immersed in a V_M are derived in this section. These results (in particular, equations (4.6), (4.7) and (4.12)) are used in the subsequent sections. The V_N is defined by the system of equations, $y^\alpha = y^\alpha(x^i)$, and V_P by $x^i = x^i(w^r)$. The first fundamental forms of V_M, V_N and V_P are (2.1), (2.2) and

$$ds^2 = h_{pq}dw^p dw^q \tag{4.1}$$

respectively. We make assumptions concerning the ranks of the functional matrices $\| y^\alpha_{,i} \|$ and $\| x^i_{,p} \|$ and the classes of $f_{\alpha\beta}(y^\gamma)$, $y^\alpha(x^i)$ and $x^i(w^p)$ similar to those in §2. Under these conditions the Gauss and Codazzi equations of the V_N in V_M and V_P in V_N and in V_M are valid. We denote the components of any $M - N$ mutually orthogonal unit vectors in V_M perpendicular to V_N by ${}_{(\sigma)}\xi^\alpha$. The components in the y's of any $N - P$ mutually orthogonal unit normals to V_P in V_N are written ${}_{(a)}\xi^\alpha$. The corresponding components in the x's of these normals are ${}_{(a)}\zeta^i$ so that

$$ {}_{(a)}\xi^\alpha = {}_{(a)}\zeta^i y^\alpha_{,i}. \tag{4.2} $$

Now it is known[15] that the following equations hold for the imbedding of the V_N in V_M, V_P in V_M and V_P in V_N respectively:

$$ \begin{aligned} y^\alpha_{,ij} &= -\begin{Bmatrix} \alpha \\ \beta\gamma \end{Bmatrix}_f y^\beta_{,i} y^\gamma_{,j} + \sum_\sigma e_\sigma \, {}_{(\sigma)}\Omega_{ij} \, {}_{(\sigma)}\xi^\alpha, \\ y^\alpha_{,pq} &= -\begin{Bmatrix} \alpha \\ \beta\gamma \end{Bmatrix}_f y^\beta_{,p} y^\gamma_{,q} + \sum_\sigma e_\sigma \, {}_{(\sigma)}\bar{\Omega}_{pq} \, {}_{(\sigma)}\xi^\alpha + \sum_a e_a \, {}_{(a)}\bar{\Omega}_{pq} \, {}_{(a)}\xi^\alpha, \\ x^i_{,pq} &= -\begin{Bmatrix} i \\ jk \end{Bmatrix}_g x^j_{,p} x^k_{,q} + \sum_a e_a \, {}_{(a)}\bar{\Omega}_{pq} \, {}_{(a)}\xi^\alpha. \end{aligned} \tag{4.3} $$

Here the $\begin{Bmatrix} \alpha \\ \beta\gamma \end{Bmatrix}_f$ and $\begin{Bmatrix} i \\ jk \end{Bmatrix}_g$ are the Christoffel symbols of the second kind formed with respect to (2.1) and (2.2) respectively and

$$ e_\sigma = f_{\alpha\beta} \, {}_{(\sigma)}\xi^\alpha \, {}_{(\sigma)}\xi^\beta, \qquad e_a = f_{\alpha\beta} \, {}_{(a)}\xi^\alpha \, {}_{(a)}\xi^\beta = g_{ij} \, {}_{(a)}\zeta^i \, {}_{(a)}\zeta_j, $$

and the Ω's are tensors from which the second fundamental forms of the spaces are constructed. Since

$$ y^\alpha_{,p} = \frac{\partial y^\alpha}{\partial x^i} \cdot x^i_{,p} \tag{4.4} $$

it follows by covariant differentiation with respect to the w's and (4.1) that

$$ y^\alpha_{,pq} = \left(y^\alpha_{,ij} + \begin{Bmatrix} k \\ ij \end{Bmatrix}_g y^\alpha_{,k} \right) x^i_{,p} x^j_{,q} + y^\alpha_{,i} x^i_{,pq}. \tag{4.5} $$

From equations (4.2) to (4.5),

$$ \sum_\sigma e_\sigma \, {}_{(\sigma)}\bar{\Omega}_{pq} \, {}_{(\sigma)}\xi^\alpha + \sum_a e_a \, {}_{(a)}\bar{\Omega}_{pq} \, {}_{(a)}\xi^\alpha = \sum_\sigma e_\sigma \, {}_{(\sigma)}\Omega_{ij} x^i_{,p} x^j_{,q} \, {}_{(\sigma)}\xi^\alpha + \sum_a e_a \, {}_{(a)}\bar{\Omega}_{pq} \, {}_{(a)}\xi^\alpha. $$

Since the normals are mutually orthogonal, these last equations give the fundamental relations

$$ {}_{(a)}\bar{\Omega}_{pq} = {}_{(a)}\bar{\Omega}_{pq}, \tag{4.6} $$

$$ {}_{(\sigma)}\bar{\Omega}_{pq} = {}_{(\sigma)}\Omega_{ij} x^i_{,p} x^j_{,q}. \tag{4.7} $$

[15] Eisenhart, *loc. cit.*, p. 160.

We now find a useful connection between the coefficients of rotation of certain orthogonal ennuples in V_N and the tensors ${}_{(a)}\bar{\Omega}_{pq}$. Let ${}_{(m)}\zeta^i$ and ${}_{(m)}\zeta_i$ be the contravariant and covariant components respectively in the x's of an orthogonal ennuple in V_N which has the property that the curves of the P congruences ${}_{(p)}\zeta^i$ which have a point in common with V_P lie in V_P. Of course, the remaining vectors ${}_{(a)}\zeta^i$ at points of V_P are orthogonal to V_P. At points of V_P, the components in the w's of ${}_{(p)}\zeta^i$ are written ${}_{(p)}\mu^q$ so that

$$ {}_{(p)}\zeta^i = {}_{(p)}\mu^q x^i_{,q}. \tag{4.8} $$

The coefficients of rotation of the ennuple are defined by

$$ \gamma_{klm} = {}_{(k)}\zeta_{i,j}\,{}_{(l)}\zeta^i\,{}_{(m)}\zeta^j. \tag{4.9} $$

The coefficients of rotation are skew-symmetric in the first two indices, that is,

$$ \gamma_{klm} + \gamma_{lkm} = 0. \tag{4.10} $$

It follows from (4.8) and (4.9) that at points of V_P,

$$ \gamma_{paq} = \left[\frac{\partial}{\partial x^j}({}_{(p)}\mu^r x^i_{,r}) + {}_{(p)}\mu^r x^k_{,r}\left\{\begin{matrix} i \\ kj \end{matrix}\right\}_a\right] {}_{(a)}\zeta_i\,{}_{(q)}\mu^s x^j_{,s}. $$

Upon simplification of these equations by means of the relation

$$ x^i_{,r}\,{}_{(a)}\zeta_i = 0, \tag{4.11} $$

the equations for the γ_{paq} become

$$ \gamma_{paq} = \left[\frac{\partial^2 x^i}{\partial w^r\,\partial w^s} + x^k_{,r}x^j_{,s}\left\{\begin{matrix} i \\ kj \end{matrix}\right\}_a\right] {}_{(p)}\mu^r\,{}_{(q)}\mu^s\,{}_{(a)}\zeta_i. $$

It follows from these equations, (4.11) and the last set of equations of (4.3) with p, q replaced by r, s that

$$ \gamma_{paq} = {}_{(a)}\bar{\Omega}_{rs}\,{}_{(p)}\mu^r\,{}_{(q)}\mu^s \tag{4.12} $$

at all points of V_P.

5. The lines of curvature of a proper V_N in an S_{N+1}

The Codazzi equations of a V_N in a V_{N+1} are

$$ \Omega_{ij,k} - \Omega_{ik,j} = \bar{R}_{\alpha\beta\gamma\delta}y^\alpha_{,i}y^\gamma_{,j}y^\delta_{,k}\xi^\beta. $$

It follows easily from the remarks following (3.11) and from Theorem 3.2 that these equations assume the form

$$ \Omega_{ij,k} - \Omega_{ik,j} = 0 \tag{5.1} $$

for every V_N if and only if V_{N+1} is an S_{N+1}. We assume this to be the case. If V_N is a proper hypersurface, the second fundamental tensor may be written as

$$ \Omega_{ij} = \sum_{m=1}^{N} e_m \rho_m\,{}_{(m)}\lambda_i\,{}_{(m)}\lambda_j, $$

where, as in §§2 and 3, the ${}_{(m)}\lambda^i$ are the tangents to the lines of curvature of V_N. If one of the ρ_m does not equal any of the others, it is called a *simple* principal normal curvature. The Codazzi equations (5.1) become

$$\sum_{m=1}^{N} e_m [\rho_m({}_{(m)}\lambda_{i,k}\,{}_{(m)}\lambda_j + {}_{(m)}\lambda_i\,{}_{(m)}\lambda_{j,k} - {}_{(m)}\lambda_{i,j}\,{}_{(m)}\lambda_k - {}_{(m)}\lambda_i\,{}_{(m)}\lambda_{k,j}) + (\rho_{m,k}\,{}_{(m)}\lambda_i\,{}_{(m)}\lambda_j - \rho_{m,j}\,{}_{(m)}\lambda_i\,{}_{(m)}\lambda_k)] = 0.$$

We multiply these last equations successively by ${}_{(n)}\lambda^i\,{}_{(l)}\lambda^j\,{}_{(m)}\lambda^k$ and ${}_{(n)}\lambda^i\,{}_{(n)}\lambda^j\,{}_{(l)}\lambda^k$ and sum for i, j and k. This gives

$$\gamma_{lnm}(\rho_n - \rho_l) = \gamma_{mnl}(\rho_n - \rho_m), \qquad (n, l, m \neq), \tag{5.2}$$

$$e_n \cdot \frac{\partial \rho_n}{\partial s_l} = (\rho_l - \rho_n)\gamma_{lnn}, \tag{5.3}$$

where $\partial\rho_n/\partial s_l$ is an intrinsic derivative and is equal to $\rho_{n,i}\,{}_{(l)}\lambda^i$. The two relations (5.2) and (5.3) will be used frequently in what follows. Since the ${}_{(m)}\lambda^i$ are an orthogonal ennuple and $\Omega_{ij,k} - \Omega_{ik,j}$ is skew-symmetric with respect to the indices j, k, it follows that (5.2) and (5.3) are equivalent to (5.1). This fact and the remarks at the beginning of this section prove

THEOREM 5.1. *The only Riemann spaces in which* (5.2) *and* (5.3) *hold for every proper hypersurface are the spaces of constant curvature.*

We now suppose that $P(1 < P < N)$ of the congruences ${}_{(r)}\lambda^i$ lie in $\infty^{N-P} V_P$ and the corresponding ρ_r do not equal any of the remaining ρ_a. Of course, the ${}_{(a)}\lambda^i$ are normal to the V_P. The equations of imbedding of the V_P are $x^i = x^i(w^p, c^a)$ where the c^a are arbitrary constants. If the components in the w's of the ${}_{(r)}\lambda^i$ are written ${}_{(r)}\eta^q$, it follows from (4.12) that

$$\gamma_{paq} = {}_{(a)}\bar{\Omega}_{rs}\,{}_{(p)}\eta^r\,{}_{(q)}\eta^s \tag{5.4}$$

throughout V_N. It follows from (5.4) that

$$\gamma_{paq} = \gamma_{qap}. \tag{5.5}$$

Indeed (5.5) is equivalent to the hypothesis that the ${}_{(a)}\lambda^i$ are orthogonal to $\infty^{N-P} V_P$ as was shown by Levy.[16] It follows from (5.5) and (5.2) with l, n, m replaced by p, a, q respectively that either

$$\gamma_{paq} = 0, \qquad (p \neq q), \tag{5.6}$$

or $\rho_p = \rho_q$, since ρ_a is different from ρ_p and ρ_q. But if $\rho_p = \rho_q$, we have from (5.2) with l, n, m replaced by a, p, q respectively that (5.6) also holds. Hence, from (5.4) and (5.6)

$${}_{(a)}\bar{\Omega}_{rs} = \sum_{q=1}^{P} \gamma_{qaq}\,{}_{(q)}\eta_r\,{}_{(q)}\eta_s. \tag{5.7}$$

[16] H. Levy, *Normal congruences of curves in Riemann space*, Bulletin of the American Mathematical Society, vol. 31 (1925), p. 41.

Thus the ${}_{(q)}\eta^r$ and ${}_{(q)}\lambda^i$ are lines of curvature of the V_P for the normals ${}_{(a)}\lambda^i$ and hence for any normal in V_N. This proves

THEOREM 5.2. *If P congruences ${}_{(q)}\lambda^i$ of lines of curvature of a proper V_N in an S_{N+1} lie in $\infty^{N-P} V_P$, where $1 < P < N$, and the corresponding ρ_q do not equal any of the remaining $N - P$ principal normal curvatures, then the ${}_{(q)}\lambda^i$ are lines of curvature for the V_P for any normal in V_N.*

For $P = N - 1$, this theorem shows that *if ${}_{(N)}\lambda^i$ is a normal congruence and ρ_N is simple, the remaining lines of curvature of V_N are also lines of curvature of the V_{N-1}'s orthogonal to ${}_{(N)}\lambda^i$.* This result may be applied to the case where ${}_{(N)}\lambda^i$ is a geodesic congruence and ρ_N is not constant. For it will be shown later[17] that in this case ${}_{(N)}\lambda^i$ is also normal.

According to (5.3) and (5.7),

$$(5.8) \qquad {}_{(a)}\bar{\Omega}_{rs} = \sum_{q=1}^{P} e_q \frac{\partial \rho_q}{\partial s_a} \Big/ (\rho_q - \rho_a) \cdot {}_{(q)}\eta_r \, {}_{(q)}\eta_s .$$

Hence *the V_P of Theorem 5.2 are totally geodesic if and only if $\partial \rho_q / \partial s_a = 0$.*

The condition[18] that a curve ${}_{(N)}\lambda^i$ of an orthogonal ennuple be a geodesic is that

$$\gamma_{lNN} = 0$$

along this curve. If C is a line of curvature corresponding to ρ_N, we note that C is orthogonal to the hypersurfaces, ρ_N = constant, if and only if $\partial \rho_N / \partial s_r = 0$ $(r = 1, 2, \cdots, N - 1)$. From this observation and the last equation and (5.3) we easily prove

THEOREM 5.3. *If C is a geodesic line of curvature of a proper V_N in an S_{N+1} corresponding to the principal normal curvature ρ_N, then either ρ_N is a constant or C is orthogonal to the hypersurfaces of V_N on which ρ_N is constant. If ρ_N is simple, the converse is true.*

The results stated above are all properties of the lines of curvature of the V_N and hence depend upon the manner in which V_N is imbedded in S_{N+1}. However, by means of Theorem 3.1, it is sometimes possible to determine geometric properties of the Ricci principal directions from the character of the lines of curvature. Such properties are *intrinsic* necessary conditions that a proper imbedding of a V_N in an S_{N+1} exist. In particular, it gives necessary conditions that a V_N be proper and of class one. As an example, we prove the following analogue of Theorem 5.2:

THEOREM 5.4. *If P congruences ${}_{(q)}\zeta^i$ of Ricci principal directions of a proper V_N in an S_{N+1} lie in $\infty^{N-P} V_P$, where $1 < P < N$, and the corresponding Ricci principal curvatures κ_q are simple, the ${}_{(q)}\zeta^i$ are also Ricci principal directions of the V_P as well as lines of curvature of the V_P for any normal in V_N.*

The proof follows. By Theorem 3.1, every line of curvature is also a Ricci principal direction of V_N. But from the concluding remarks of §3, it is seen

---

[17] Cf. Theorem 5.3 infra.

[18] Eisenhart, loc. cit., p. 100.

that, conversely, the ${}_{(q)}\zeta^i$ are also lines of curvature ${}_{(q)}\lambda^i$ of V_N and that the remaining lines of curvature ${}_{(a)}\lambda^i$ of V_N lie in the linear vector space determined by the Ricci principal directions ${}_{(a)}\zeta^i$ at each point. Since the ${}_{(q)}\zeta^i (\equiv {}_{(q)}\lambda^i)$ lie in $\infty^{N-P} V_P$, it follows from Theorem 5.2 that the ${}_{(q)}\zeta^i$ are lines of curvature of the V_P for any normal in V_N.

It remains to show that the ${}_{(q)}\zeta^i$ are Ricci principal directions of the V_P. The Gauss equations for any V_P in V_N are

$$\tilde{R}_{pqrs} = \sum_{a=P+1}^{N} e_a({}_{(a)}\bar{\Omega}_{pr}\,{}_{(a)}\bar{\Omega}_{qs} - {}_{(a)}\bar{\Omega}_{ps}\,{}_{(a)}\bar{\Omega}_{qr}) + R_{ijkl}x^i_{,p}x^j_{,q}x^k_{,r}x^l_{,s} \tag{5.9}$$

where $\tilde{R}_{pqrs}$ is the Riemann curvature tensor of V_P. Since the ${}_{(a)}\bar{\Omega}_{rs}$ satisfy (5.7), a coördinate system $\{w^p\}$ exists at any point A of V_P such that the coordinate directions are tangent to the ${}_{(p)}\lambda^i$ and

$$\begin{aligned} &h_{pp} = e_p\,, \qquad h_{pq} = 0, \\ &{}_{(a)}\Omega_{pq} = 0, \end{aligned} \qquad (p \neq q;\ a = P+1, P+2, \cdots, N). \tag{5.10}$$

From (5.9) and (5.10), we obtain at A in the coördinate system $\{w^p\}$

$$\tilde{R}_{qr} = R_{ijkl}x^i_{,p}x^j_{,q}x^k_{,r}x^l_{,s}h^{ps}, \qquad (q \neq r), \tag{5.11}$$

where $\tilde{R}_{qr} = h^{ps}\tilde{R}_{pqrs}$ is the Ricci tensor of V_P. Proceeding as in the derivation of (3.9), we find

$$x^i_{,p}x^l_{,s}h^{ps} = g^{il} - \sum_{a=P+1}^{N} e_a\,{}_{(a)}\lambda^i\,{}_{(a)}\lambda^l.$$

Hence (5.11) becomes

$$\tilde{R}_{qr} = R_{jk}\,{}_{(q)}\lambda^j\,{}_{(r)}\lambda^k - \sum_{a=P+1}^{N} e_a R_{ijkl}\,{}_{(a)}\lambda^i\,{}_{(q)}\lambda^j\,{}_{(r)}\lambda^k\,{}_{(a)}\lambda^l, \qquad (q \neq r), \tag{5.12}$$

since $x^i_{,q} = {}_{(q)}\lambda^i$ in the coördinate system $\{w^p\}$.

The Gauss equations for V_N in S_{N+1}

$$R_{ijkl} = e(\Omega_{ik}\Omega_{jl} - \Omega_{il}\Omega_{jk}) + K_0(g_{ik}g_{jl} - g_{il}g_{jk}) \tag{5.13}$$

show that

$$R_{ijkl}\,{}_{(a)}\lambda^i\,{}_{(q)}\lambda^j\,{}_{(r)}\lambda^k\,{}_{(a)}\lambda^l = 0, \tag{5.14}$$

since $\Omega_{ij(m)}\lambda^i\,{}_{(n)}\lambda^j = 0$ when $m \neq n$. By hypothesis ${}_{(q)}\lambda^i$ and ${}_{(r)}\lambda^i$ are Ricci principal directions of V_N. Hence

$$R_{jk}\,{}_{(q)}\lambda^j\,{}_{(r)}\lambda^k = 0. \tag{5.15}$$

It follows from (5.12), (5.14) and (5.15) that

$$\tilde{R}_{qr} = 0, \qquad (q \neq r),$$

which shows that the coördinate directions ${}_{(q)}\lambda^i$ are Ricci principal directions of V_P.

Thus the ${}_{(q)}\eta^r$ and ${}_{(q)}\lambda^i$ are lines of curvature of the V_P for the normals ${}_{(a)}\lambda^i$ and hence for any normal in V_N. This proves

THEOREM 5.2. *If P congruences ${}_{(q)}\lambda^i$ of lines of curvature of a proper V_N in an S_{N+1} lie in $\infty^{N-P} V_P$, where $1 < P < N$, and the corresponding ρ_q do not equal any of the remaining $N - P$ principal normal curvatures, then the ${}_{(q)}\lambda^i$ are lines of curvature for the V_P for any normal in V_N.*

For $P = N - 1$, this theorem shows that *if ${}_{(N)}\lambda^i$ is a normal congruence and ρ_N is simple, the remaining lines of curvature of V_N are also lines of curvature of the V_{N-1}'s orthogonal to ${}_{(N)}\lambda^i$.* This result may be applied to the case where ${}_{(N)}\lambda^i$ is a geodesic congruence and ρ_N is not constant. For it will be shown later[17] that in this case ${}_{(N)}\lambda^i$ is also normal.

According to (5.3) and (5.7),

$$(5.8) \qquad {}_{(a)}\bar{\Omega}_{rs} = \sum_{q=1}^{P} e_q \frac{\partial \rho_q}{\partial s_a} \Big/ (\rho_q - \rho_a) \cdot {}_{(q)}\eta_r \, {}_{(q)}\eta_s .$$

Hence *the V_P of Theorem 5.2 are totally geodesic if and only if $\partial \rho_q / \partial s_a = 0$.*

The condition[18] that a curve ${}_{(N)}\lambda^i$ of an orthogonal ennuple be a geodesic is that

$$\gamma_{\iota NN} = 0$$

along this curve. If C is a line of curvature corresponding to ρ_N, we note that C is orthogonal to the hypersurfaces, $\rho_N =$ constant, if and only if $\partial \rho_N / \partial s_r = 0$ $(r = 1, 2, \cdots, N - 1)$. From this observation and the last equation and (5.3) we easily prove

THEOREM 5.3. *If C is a geodesic line of curvature of a proper V_N in an S_{N+1} corresponding to the principal normal curvature ρ_N, then either ρ_N is a constant or C is orthogonal to the hypersurfaces of V_N on which ρ_N is constant. If ρ_N is simple, the converse is true.*

The results stated above are all properties of the lines of curvature of the V_N and hence depend upon the manner in which V_N is imbedded in S_{N+1}. However, by means of Theorem 3.1, it is sometimes possible to determine geometric properties of the Ricci principal directions from the character of the lines of curvature. Such properties are *intrinsic* necessary conditions that a proper imbedding of a V_N in an S_{N+1} exist. In particular, it gives necessary conditions that a V_N be proper and of class one. As an example, we prove the following analogue of Theorem 5.2:

THEOREM 5.4. *If P congruences ${}_{(q)}\zeta^i$ of Ricci principal directions of a proper V_N in an S_{N+1} lie in $\infty^{N-P} V_P$, where $1 < P < N$, and the corresponding Ricci principal curvatures κ_q are simple, the ${}_{(q)}\zeta^i$ are also Ricci principal directions of the V_P as well as lines of curvature of the V_P for any normal in V_N.*

The proof follows. By Theorem 3.1, every line of curvature is also a Ricci principal direction of V_N. But from the concluding remarks of §3, it is seen

---

[17] Cf. Theorem 5.3 infra.

[18] Eisenhart, loc. cit., p. 100.

that, conversely, the ${}_{(q)}\zeta^i$ are also lines of curvature ${}_{(q)}\lambda^i$ of V_N and that the remaining lines of curvature ${}_{(a)}\lambda^i$ of V_N lie in the linear vector space determined by the Ricci principal directions ${}_{(a)}\zeta^i$ at each point. Since the ${}_{(q)}\zeta^i (\equiv {}_{(q)}\lambda^i)$ lie in ${}^{\infty N-P}V_P$, it follows from Theorem 5.2 that the ${}_{(q)}\zeta^i$ are lines of curvature of the V_P for any normal in V_N.

It remains to show that the ${}_{(q)}\zeta^i$ are Ricci principal directions of the V_P. The Gauss equations for any V_P in V_N are

$$\tilde{R}_{pqrs} = \sum_{a=P+1}^{N} e_a({}_{(a)}\bar{\Omega}_{pr}\,{}_{(a)}\bar{\Omega}_{qs} - {}_{(a)}\bar{\Omega}_{ps}\,{}_{(a)}\bar{\Omega}_{qr}) + R_{ijkl}x^i_{,p}x^j_{,q}x^k_{,r}x^l_{,s} \tag{5.9}$$

where $\tilde{R}_{pqrs}$ is the Riemann curvature tensor of V_P. Since the ${}_{(a)}\bar{\Omega}_{rs}$ satisfy (5.7), a coördinate system $\{w^p\}$ exists at any point A of V_P such that the coordinate directions are tangent to the ${}_{(p)}\lambda^i$ and

$$\begin{aligned} &h_{pp} = e_p\,, \qquad h_{pq} = 0, \\ &{}_{(a)}\Omega_{pq} = 0, \end{aligned} \qquad (p \neq q;\ a = P+1, P+2, \cdots, N). \tag{5.10}$$

From (5.9) and (5.10), we obtain at A in the coördinate system $\{w^p\}$

$$\tilde{R}_{qr} = R_{ijkl}x^i_{,p}x^j_{,q}x^k_{,r}x^l_{,s}h^{ps}, \qquad (q \neq r), \tag{5.11}$$

where $\tilde{R}_{qr} = h^{ps}\tilde{R}_{pqrs}$ is the Ricci tensor of V_P. Proceeding as in the derivation of (3.9), we find

$$x^i_{,p}x^l_{,s}h^{ps} = g^{il} - \sum_{a=P+1}^{N} e_a\,{}_{(a)}\lambda^i\,{}_{(a)}\lambda^l.$$

Hence (5.11) becomes

$$\tilde{R}_{qr} = R_{jk}\,{}_{(q)}\lambda^j\,{}_{(r)}\lambda^k - \sum_{a=P+1}^{N} e_a R_{ijkl}\,{}_{(a)}\lambda^i\,{}_{(q)}\lambda^j\,{}_{(r)}\lambda^k\,{}_{(a)}\lambda^l, \qquad (q \neq r), \tag{5.12}$$

since $x^i_{,q} = {}_{(q)}\lambda^i$ in the coördinate system $\{w^p\}$.

The Gauss equations for V_N in S_{N+1}

$$R_{ijkl} = e(\Omega_{ik}\Omega_{jl} - \Omega_{il}\Omega_{jk}) + K_0(g_{ik}g_{jl} - g_{il}g_{jk}) \tag{5.13}$$

show that

$$R_{ijkl}\,{}_{(a)}\lambda^i\,{}_{(q)}\lambda^j\,{}_{(r)}\lambda^k\,{}_{(a)}\lambda^l = 0, \tag{5.14}$$

since $\Omega_{ij(m)}\lambda^i\,{}_{(n)}\lambda^j = 0$ when $m \neq n$. By hypothesis ${}_{(q)}\lambda^i$ and ${}_{(r)}\lambda^i$ are Ricci principal directions of V_N. Hence

$$R_{jk}\,{}_{(q)}\lambda^j\,{}_{(r)}\lambda^k = 0. \tag{5.15}$$

It follows from (5.12), (5.14) and (5.15) that

$$\tilde{R}_{qr} = 0, \qquad (q \neq r),$$

which shows that the coördinate directions ${}_{(q)}\lambda^i$ are Ricci principal directions of V_P.

6. Several equal principal normal curvatures

We now consider the case where $P(1 < P < N)$ and only P of the principal normal curvatures are equal to the same point function, i.e.,

$$\rho_q = \rho. \tag{6.1}$$

All points of such a V_N are partially umbilical. It follows from (5.2) with l, n, m replaced by a, p, q and (4.10) that (5.6) is true. Hence, the conditions (5.5) that the ${}_{(r)}\lambda^i$ lie in $\infty^{N-P} V_P$ are satisfied. Since (5.8) is valid in this case and the metric tensor h_{rs} may always be written as

$$h_{rs} = \sum_{q=1}^{P} e_q \, {}_{(q)}\eta_r \, {}_{(q)}\eta_s \tag{6.2}$$

it follows from (5.8) and (6.1) that

$${}_{(a)}\bar{\Omega}_{rs} = \frac{\partial \rho}{\partial s_a} \Big/ (\rho - \rho_a) \cdot h_{rs}. \tag{6.3}$$

Hence the V_P are umbilical in V_N. They are totally geodesic in V_N if and only if

$$\frac{\partial \rho}{\partial s_a} = 0. \tag{6.4}$$

But we find from (6.1) and (5.3) with l, n replaced by p, q respectively that

$$\frac{\partial \rho}{\partial s_p} = 0. \tag{6.5}$$

Hence (6.4) is equivalent to $\rho =$ constant. Also, from (6.5), we deduce that ρ is constant on each V_P.

According to (6.1), the coefficients of the second fundamental form of V_N in S_{N+1} may be written as

$${}_{(N+1)}\Omega_{ij} \equiv \Omega_{ij} = \sum_{q=1}^{P} e_q \rho \, {}_{(q)}\lambda_i \, {}_{(q)}\lambda_j + \sum_{a=P+1}^{N} e_a \rho_a \, {}_{(a)}\lambda_i \, {}_{(a)}\lambda_j. \tag{6.6}$$

From (4.6) and (6.3) we have for the V_P considered as subspaces of S_{N+1}

$${}_{(a)}\tilde{\Omega}_{rs} = \frac{\partial \rho}{\partial s_a} \Big/ (\rho - \rho_a) \cdot h_{rs}, \tag{6.7}$$

and from (4.7), (4.11), (6.2) and (6.6)

$${}_{(N+1)}\tilde{\Omega}_{rs} = \rho h_{rs}. \tag{6.8}$$

Hence the V_P are umbilical in S_{N+1}. They are totally geodesic in S_{N+1} if and only if $\rho = 0$. The Gauss equations of the V_P in S_{N+1} become

$$\tilde{R}_{pqrs} = \left(\sum_{a=P+1}^{N} e_a \left(\frac{\partial \rho}{\partial s_a} \Big/ (\rho - \rho_a) \right)^2 + e\rho^2 + K_0 \right) (h_{pr} h_{qs} - h_{ps} h_{qr}) \tag{6.9}$$

after account is taken of (6.7) and (6.8). Therefore, if $P > 2$ or ρ is a constant, the V_P are spaces of constant curvature. This proves

THEOREM 6.1. *Let P and only $P (1 < P < N)$ of the principal normal curvatures of a proper V_N in an S_{N+1} be equal to the same point function ρ. Then the lines of curvature corresponding to ρ lie in $\infty^{N-P} V_P$ which are umbilical in S_{N+1}. If $P > 2$, the V_P are S_P. The value of ρ is constant on each V_P. The V_P are totally geodesic S_P in V_N if and only if ρ is a constant. They are totally geodesic in S_{N+1} if and only if $\rho = 0$.*

Some of the results stated in this theorem and Theorem 5.3 were obtained previously by Struik.[19] As a special case of the above theorem, let $P = N - 1$ and $\rho = 0$. It follows that the corresponding V_N contains $\infty^1 S_{N-1}$ which are totally geodesic in S_{N+1}. It is known[20] that these V_N's are the developable hypersurfaces of S_{N+1}.

The second fundamental form of the developable V_N in S_{N+1} is

$$\Omega_{ij} = e_N \rho_{N(N)} \lambda_{i(N)} \lambda_j . \tag{6.10}$$

If the S_{N+1} is defined by[21]

$$e_1 z^{1^2} + \cdots + e_{N+2} z^{N+2^2} = \frac{1}{K_0},$$

where the e's are $+1$ or -1, the variation of the normal ξ^τ to V_N in S_{N+1} is given by[22]

$$\xi^\tau_{.j} = -\Omega_{ij} g^{ik} z^\tau_{,k} , \qquad (\tau = 1, 2, \cdots, N+2). \tag{6.11}$$

From (6.10) and (6.11),

$$\xi^\tau_{.j\ (p)} \lambda^j = 0.$$

Since the $N - 1$ orthogonal vectors ${}_{(p)}\lambda^j$ lie in the S_{N-1}'s of V_N, it follows that the normal to V_N is displaced parallel to itself for all directions in any fixed S_{N-1}.

If S_N is the osculating geodesic hypersurface of V_N at a point A lying in an S_{N-1} of V_N, S_N contains S_{N-1}. Since the normal to S_N is also displaced parallel to itself and coincides with ξ^τ at A, it follows that normals to S_N and V_N coincide at all points of S_{N-1}. Hence the osculating S_N is the same at all points of S_{N-1}. This is the well known characteristic property of developable hypersurfaces.

We now apply the results of Theorem 6.1 to a special case which arises in the

---

[19] Struik, *loc. cit.*, p. 142.

[20] Struik, *loc. cit.*, pp. 126, 144 and C. Burstin, *Beiträge zur mahrdimensionalen Differentialgeometrie*, Monatshefte für Mathematik und Physik, vol. 36 (1929), p. 98. Levy apparently overlooked these hypersurfaces in his theorem that the lines of curvature of an S_N in an S_{N+1} are indeterminate (cf. H. Levy, *Tensors Determined by a Hypersurface in a Riemann Space*, Transactions of the American Mathematical Society, vol. 28 (1926), p. 693).

[21] If $K_0 = 0$, we use the cartesian coördinates of the flat S_{N+1} to obtain equations analogous to (6.11).

[22] Eisenhart, *loc. cit.*, p. 211.

classification of Einstein spaces E_N in an S_{N+1}. We suppose that two of the principal normal curvatures of a proper V_N in an S_{N+1} are constant and neither is simple. Let ρ and ρ' be the two unequal constant curvatures so that[23]

$$\begin{aligned} \rho_q &= \rho\,, \\ \rho_\epsilon &= \rho'. \end{aligned} \tag{6.12}$$

It follows from Theorem 6.1 that the ${}_{(q)}\lambda^i$ and ${}_{(\epsilon)}\lambda^i$ lie in $\infty^{N-P}S_P$ and $\infty^{N-Q+P}S_{Q-P}$ respectively which are geodesic in V_N. From (6.9), the S_P and S_{Q-P} have Riemannian curvatures

$$K_1 = e\rho^2 + K_0 \tag{6.13}$$

and

$$K_2 = e\rho'^2 + K_0 \tag{6.14}$$

respectively. We shall later show the existence of a coördinate system $\{x^i\}$ in which

$$g_{q\epsilon}(x^i) \equiv 0 \tag{6.15}$$

and the equations of the S_P and S_{Q-P} become

$$x^a = c^a \tag{6.16}$$

and

$$x^\nu = c^\nu \tag{6.17}$$

respectively. Hence the first fundamental form of V_N is[24]

$$ds^2 = g_{pq}(x^r)dx^p dx^q + g_{\epsilon\theta}(x^\phi)dx^\epsilon dx^\theta + g_{\omega i}(x^j)dx^\omega dx^i \tag{6.18}$$

where $g_{pq}(x^r)$ and $g_{\epsilon\theta}(x^\phi)$ are the first fundamental forms of an S_P of Riemann curvature K_1 and an S_{Q-P} of Riemann curvature K_2 respectively. According to (6.12), the coefficients of the second fundamental form of V_N in S_{N+1} satisfy

$$\begin{aligned} \Omega_{pq} &= \rho g_{pq}\,, \qquad \Omega_{\epsilon\theta} = \rho' g_{\epsilon\theta}\,, \\ \Omega_{\epsilon p} &= 0, \end{aligned} \tag{6.19}$$

in the coördinate system $\{x^i\}$.

---

[23] The indices ϵ, θ, ϕ; ν; ω have the ranges $P+1, P+2, \cdots, Q$; $1, 2, \cdots, P, Q+1, Q+2, \cdots, N$; $Q+1, Q+2, \cdots, N$ respectively. The ranges of the other indices are as before. It is assumed that $2 < P+1 < Q \leqq N$.

[24] When the first fundamental form of a V_N is

$$ds^2 = g_{pq}(x^r)dx^p dx^q + g_{ab}(x^i)dx^a dx^b,$$

where $g_{pq}(x^r)$ are functions of the variables x^r only, the subspaces, x^a = constant, are totally geodesic in V_N. Cf. E. Bompiani, *Spazi Riemanniani luoghi di varietà totalmente geodetiche*, Rendiconti di Palermo, vol. 48 (1924), p. 124.

From (6.18), by direct calculation we find that

$$R_{\epsilon\rho\theta_Q} = 0.$$

It follows from the Gauss equations (5.13) for V_N and from (6.19) that

$$e\rho\rho' + K_0 = 0. \tag{6.20}$$

We consider two cases:

(i) $K_0 \neq 0$. According to (6.20), neither ρ nor ρ' can be zero and ρ' is uniquely defined by ρ. Hence V_N can have at most two different constant non-simple principal normal curvatures. From (6.13), (6.14) and (6.20), the corresponding Riemann curvatures of S_P, S_{Q-P} and V_N are all different from zero and satisfy

$$\frac{1}{K_1} + \frac{1}{K_2} = \frac{1}{K_0}. \tag{6.21}$$

If all the principal normal curvatures are constant and none are simple, it follows that $Q = N$ and the first fundamental form (6.18) of V_N is

$$ds^2 = \frac{e_1(dx^1)^2 + \cdots + e_P(dx^P)^2}{\left[1 + \frac{K_1}{4}(e_1x^{1^2} + \cdots + e_Px^{P^2})\right]^2} + \frac{e_{P+1}(dx^{P+1})^2 + \cdots + e_N(dx^N)^2}{\left[1 + \frac{K_2}{4}(e_{P+1}x^{P+1^2} + \cdots + e_Nx^{N^2})\right]^2}, \tag{6.22}$$

where each e is $+1$ or -1. Conversely, a V_N defined by (6.22) may be imbedded in an S_{N+1} whose Riemannian curvature is given by (6.21) by means of the algebraic equations

$$\begin{aligned} e_1z^{1^2} + \cdots + e_{P+1}z^{P+1^2} &= \frac{1}{K_1}, \\ e_{P+2}z^{P+2^2} + \cdots + e_{N+2}z^{N+2^2} &= \frac{1}{K_2}, \end{aligned} \qquad (K_1K_2K_0 \neq 0), \tag{6.23}$$

where the e's are $+1$ or -1 and the S_{N+1} is defined by

$$e_1z^{1^2} + \cdots + e_{N+2}z^{N+2^2} = \frac{1}{K_0}.$$

The z's are the point coördinates of Weierstrass.[25] Since $N \geqq 4$ and $\rho\rho' \neq 0$, the V_N is indeformable in S_{N+1}. The equations (6.23) show that V_N may also be considered as the intersection of two hypercylinders in a flat space of $N + 2$ dimensions.

(ii) $K_0 = 0$. According to (6.20), there cannot be two different constant non-zero principal normal curvatures which are not simple. If $\rho = 0$, it follows

[25] Eisenhart, *loc. cit.*, pp. 204–207.

from (6.13) that $K_1 = 0$. If all the principal normal curvatures are constant and none are simple, $Q = N$ and the first fundamental form of V_N is (6.22) with $K_1 = 0$. This V_N may be imbedded in a flat space of $N + 1$ dimensions by means of the equations

$$z^q = x^q$$

$$e_{P+1}z^{P+1^2} + \cdots + e_{N+1}z^{N+1^2} = \frac{1}{K_2}, \qquad (K_2 \neq 0), \tag{6.24}$$

where the e's are $+1$ or -1 and the z's are cartesian coördinates of the flat space. Thus V_N is a spherical hypercylinder of the flat space. It is indeformable if $N - P > 2$. Some of the above results are stated in

THEOREM 6.2. *If the principal normal curvatures of a proper V_N in an S_{N+1} of Riemann curvature K_0 are constant, not all equal and none are simple, then P of the principal normal curvatures have one value and the remaining $N - P$ have another value* (P *is any integer so that* $1 < P < N - 1$.) *V_N contains $\infty^{N-P}S_P$ of Riemann curvature K_1 and, orthogonal to the S_P, $\infty^P S_{N-P}$ of Riemann curvature K_2. If $K_0 \neq 0$, $\frac{1}{K_1} + \frac{1}{K_2} = \frac{1}{K_0}$ and if $K_0 = 0$, $K_1 = 0$ and $K_2 \neq 0$. The S_P and S_{N-P} are totally geodesic in V_N. Conversely, each V_N of this kind can be imbedded in an S_{N+1}. The first fundamental form of V_N is given by* (6.22) *and V_N is imbedded in S_{N+1} by means of the algebraic equations* (6.23) *or* (6.24).

In order to complete the proof of this theorem, the existence of the coördinate system $\{x^i\}$ in which (6.15), (6.16) and (6.17) hold is now proved. From (6.12), as was shown in the proof of Theorem 6.1, it follows that

$$\gamma_{paq} = 0, \qquad (p \neq q),$$

$$\gamma_{\epsilon v\theta} = 0, \qquad (\epsilon \neq \theta).$$

But these relations satisfy the conditions[26] that the systems of equations

$$({}_{(p)}\lambda^i)\frac{\partial f}{\partial x^i} = 0 \tag{6.25}$$

and

$$({}_{(\epsilon)}\lambda^i)\frac{\partial g}{\partial x^i} = 0 \tag{6.26}$$

each form a complete system. Let $f^{P+1}, f^{P+2}, \cdots, f^N$ and $g^1, g^2, \cdots, g^P, g^{Q+1}, g^{Q+2}, \cdots, g^N$ denote independent solutions of (6.25) and (6.26) respectively. It is clear that $f^a = c^a$ and $g^v = c^v$ where the c's are arbitrary constants are the equations of the S_P and S_{Q-P} which contain the ${}_{(p)}\lambda^i$ and ${}_{(\epsilon)}\lambda^i$.

By a coördinate transformation, we may write the solutions of (6.26) as

$$g^v = x^v. \tag{6.27}$$

[26] Eisenhart, *loc. cit.*, p. 125 ex. 7 or the first citation to Levy.

If R of the f^a are independent of these solutions, we may write these as

$$f^{P+1} = x^{P+1}, \quad f^{P+2} = x^{P+2}, \cdots, \quad f^R = x^R.$$

Of course, $R \leqq Q$ else there would be more than N independent variables. We shall show that $R = Q$. The remaining independent solutions of (6.25) may be chosen as functions of the x^ν only. Hence for these solutions (6.25) becomes

$$(6.28) \qquad {}_{(p)}\lambda^\nu \frac{\partial f}{\partial x^\nu} = 0.$$

Now, from (6.26) and (6.27), we have

$$(6.29) \qquad {}_{(\epsilon)}\lambda^\nu = 0.$$

Since the matrices $\left\| \begin{matrix} {}_{(p)}\lambda^i \\ {}_{(\epsilon)}\lambda^i \end{matrix} \right\|$ and $\| {}_{(\epsilon)}\lambda^i \|$ have ranks Q and $Q - P$ respectively, it follows from (6.29) that the rank of $\| {}_{(p)}\lambda^\nu \|$ is P. Therefore there are $N - Q$ independent solutions $f^\omega(x^\nu)$ of (6.28), i.e., of (6.25), which are functions of the x^ν. Since the $f^\omega(x^\nu)$ are also solutions of (6.26), by means of a coördinate transformation of the x^ν, the $f^\omega(x^\nu)$ may be written as $x^{Q+1}, x^{Q+2}, \cdots, x^N$. Hence $R = Q$ and the independent solutions of (6.25) may be chosen as

$$(6.30) \qquad f^a = x^a.$$

In this coördinate system $\{x^i\}$, (6.16) and (6.17) hold.

Now from (6.25) and (6.30),

$$(6.31) \qquad {}_{(p)}\lambda^a = 0.$$

By hypothesis, $g_{ij}\,{}_{(\epsilon)}\lambda^i\,{}_{(p)}\lambda^j = 0$. It follows from (6.29) and (6.31) that $g_{\theta q}\,{}_{(\epsilon)}\lambda^\theta\,{}_{(p)}\lambda^q = 0$. Since $|\,{}_{(\epsilon)}\lambda^\theta\,| \neq 0$, $|\,{}_{(p)}\lambda^q\,| \neq 0$, this last equation is equivalent to (6.15). This completes the proof of Theorem 6.2.

7. Einstein spaces E_N in an S_{N+1}

Some of the results of the preceding section are now applied to obtain a complete classification of the proper Einstein spaces E_N in an S_{N+1}. Most of these results were previously announced without proof.[27] An E_N is defined as a Riemann space V_N whose mean curvature κ is a constant at each point, that is,

$$(7.1) \qquad R_{ij} = -\kappa g_{ij}.$$

If $N > 2$, κ is a constant throughout the space. The case $N = 2$ presents no problem since every V_2 is also an E_2. Neither do we consider the case $N = 3$. For every E_3 is an S_3 and the possibility and manner of its imbedding in a given S_4 is well known. We therefore suppose $N \geqq 4$.

We note that the assumption that E_N be proper is really restrictive. For Einstein spaces E_4 with $\kappa = 0$ exist whose first fundamental form is indefinite

[27] A. Fialkow, *Einstein spaces in a space of constant curvature*, Proceedings of the National Academy of Sciences, January 1938, pp. 30–34.

which may be imbedded in a flat space S_5 so that all the lines of curvature are tangent to null vectors. Thus, for example, the E_4 whose first fundamental form is[28]

$$ds^2 = 4[dx^1 dx^2 + dx^3 dx^4 + (x^1 dx^1 + x^3 dx^3)^2]$$

may be imbedded in the flat S_5 whose first fundamental form is

$$ds^2 = dy^{1^2} - dy^{2^2} + dy^{3^2} - dy^{4^2} + dy^{5^2}$$

by means of the equations

$$y^1 = x^1 + x^2, \quad y^3 = x^3 + x^4, \quad y^5 = x^{1^2} + x^{3^2}.$$
$$y^2 = x^1 - x^2, \quad y^4 = x^3 - x^4,$$

It is easy to verify that the tangents to the lines of curvature are all null vectors in the pencil determined by the null vectors having components (0, 1, 0, 0) and (0, 0, 0, 1).

We begin the classification of proper E_N's in an S_{N+1} by recalling a necessary condition on the first fundamental forms of S_{N+1} and E_N. If s is the signature of the first fundamental form of E_N, it is readily seen[29] that the signature of the first fundamental form of S_{N+1} must be $s + e$ where, as usual, e is defined by (2.5). We assume that this condition is satisfied in what follows.

Since $\kappa_m = \kappa$ for an E_N, (3.20) becomes

$$\kappa = e\rho_m \sum_{k \neq m} \rho_k + (N-1)K_0. \tag{7.2}$$

We solve these equations for the ρ_m. There are several cases to consider. One solution is obtained when all the ρ_m are equal to ρ. From (7.2), we find

$$\rho^2 = e\left(\frac{\kappa}{N-1} - K_0\right). \tag{7.3}$$

Since ρ must be real, this is a solution only if

(a) $$e\kappa > e(N-1)K_0$$

or

(b) $$\kappa = (N-1)K_0.$$

Since (7.2) is equivalent to

$$\rho_1 \sum_{k \neq 1} \rho_k = \rho_2 \sum_{k \neq 2} \rho_k = \cdots = \rho_N \sum_{k \neq N} \rho_k = e[\kappa - (N-1)K_0]$$

---

[28] This E_4 is a member of a set of E_4's of class one which were studied by Brinkmann in another connection. Any non-flat E_4 of this set must be an improper hypersurface of a flat S_5 in accordance with Theorem 7.3. Cf. H. W. Brinkmann, *Conformal mapping of Einstein spaces*, Mathematische Annalen, vol. 94 (1925), pp. 140–141.

[29] For example, cf. Eisenhart, *loc. cit.*, p. 188. A similar result applies to any V_N in a V_{N+1}. If the first fundamental form of V_{N+1} is positive definite, $s = N$ and $e = +1$.

if two of the ρ_m are unequal (e.g., $\rho_1 \neq \rho_N$) it follows that

$$\rho_2 + \rho_3 + \cdots + \rho_{N-1} = 0, \qquad \rho_1\rho_N = e[\kappa - (N-1)K_0], \tag{7.4}$$

$$\rho_t^2 - (\rho_1 + \rho_N)\rho_t + \rho_1\rho_N = 0 \qquad (t = 2, 3, \cdots, N-1). \tag{7.5}$$

According to (7.5), each ρ_t is equal to either ρ_1 or ρ_N. If P of the principal normal curvatures, $\rho_1, \rho_2, \cdots, \rho_P$ have the value ρ and the remaining $N - P$ principal normal curvatures $\rho_{P+1}, \rho_{P+2}, \cdots, \rho_N$ have the value ρ', it follows from (7.4) that

$$\begin{aligned} &(P-1)\rho + (N-P-1)\rho' = 0, \\ &\rho^2 = \frac{(N-P-1)e}{P-1}[(N-1)K_0 - \kappa], \qquad (1 < P < N-1), \end{aligned} \tag{7.6}$$

if $\kappa - (N-1)K_0 \neq 0$ and

$$\begin{aligned} &P = N-1, \\ &\rho = 0, \qquad \rho' \neq 0, \end{aligned} \tag{7.7}$$

if (b) is satisfied. Since ρ is real in (7.6), this solution exists only if

$$e\kappa < e(N-1)K_0. \tag{c}$$

We call an E_N an *Einstein space of type* (*a*), (*b*) or (*c*) according as it satisfies (a), (b) or (c). The above discussion proves that a proper Einstein space of type (a) in an S_{N+1} has indeterminate lines of curvature and hence is a hypersphere whose normal curvature is given by (7.3). If K is the constant Riemann curvature of the hypersphere, it follows easily from (a) that $eK > eK_0$. If E_N is of type (b), it follows from (7.3), (7.7) and (6.10) that E_N is either a totally geodesic or developable hypersurface of S_{N+1}. Its Riemann curvature is also K_0.

If (c) is satisfied, E_N has two unequal principal normal curvatures ρ and ρ' at each point. Since the normal curvatures are defined by (2.7) and g_{ij} and Ω_{ij} are continuous functions of the x's, it follows that both ρ and ρ' are continuous point functions. But there are only a finite number of solutions of (7.6) for ρ and ρ' and these solutions are distinct and constant at all points of the space. Hence ρ and ρ' are constant throughout E_N and the results of Theorem 6.2 apply. From (6.20), (7.2) and the first equation of (7.6), we find that

$$\kappa = (N-2)K_0. \tag{7.8}$$

If this value of κ is substituted in (c), it follows that

$$eK_0 > 0. \tag{7.9}$$

Also E_N contains $\infty^{N-P}S_P$ of Riemann curvature K_1 and $\infty^{P}S_{N-P}$ of Riemann curvature K_2 which are totally geodesic in E_N. From (6.13), (6.14), (7.6) and (7.8),

$$K_1 = \frac{N-2}{P-1}K_0,$$

$$K_2 = \frac{N-2}{N-P-1}K_0.$$

It is easy to see that if either K_1 or K_2 does not satisfy these relations the resulting V_N is not an E_N. Thus the Einstein spaces of type (c) are the only E_N's containing $\infty^{N-P}S_P$ and $\infty^{P}S_{N-P}$ which are totally geodesic in E_N. We state these results in

THEOREM 7.1. *Let $E_N (N \geqq 4)$ be a real Einstein space of mean curvature κ which is a proper hypersurface of a real space S_{N+1} of constant curvature K_0. Let the signatures of the first fundamental forms of E_N and S_{N+1} be s and $s + e$ respectively. If*

(a) $e\kappa > e(N-1)K_0$, then E_N is an S_N of Riemann curvature K where $eK > eK_0$. Conversely, each E_N of type (a) can be imbedded as a hypersphere of S_{N+1} and has indeterminate lines of curvature and constant normal curvature.

(b) $\kappa = (N-1)K_0$, then E_N is an S_N of Riemann curvature K_0. Each E_N of type (b) can be imbedded either as a totally geodesic or as a developable hypersurface of S_{N+1}.

(c) $e\kappa < e(N-1)K_0$, then $eK_0 > 0$ and $\kappa = (N-2)K_0$. E_N contains $\infty^{N-P}S_P$ of Riemann curvature $\frac{N-2}{P-1}K_0$ and also, orthogonal to the S_P, $\infty^{P}S_{N-P}$ of Riemann curvature $\frac{N-2}{N-P-1}K_0$. (P is any integer such that $1 < P < N$.) The S_P and S_{N-P} are totally geodesic in E_N. Each E_N of type (c) can be imbedded in S_{N+1}. The principal normal curvature of E_N has one constant value on all the S_P and another on all the S_{N-P}.

The result stated in (a) was recently proved by Thomas[30] and Cartan[31] subject to the restriction that S_{N+1} is a Euclidean space. The theorem in (b) was proved by Kasner[32] for a Euclidean S_5 and for a Euclidean space of any dimensionality by Schouten and Struik.[33] These writers were also in possession of the essential facts for type (a). Eisenhart[34] showed that the result (b) is true in

---

[30] T. Y. Thomas, *On closed spaces of constant mean curvature*, American Journal of Mathematics, vol. 58 (1936), pp. 702–704.

[31] T. Y. Thomas, *Extract from a letter from E. Cartan concerning my note: On closed spaces of constant mean curvature*, American Journal of Mathematics, vol. 59 (1937), pp. 793–794.

[32] E. Kasner, *The impossibility of Einstein fields immersed in a flat space of five dimensions*, American Journal of Mathematics, vol. 43 (1921), p. 126.

[33] Schouten and Struik, *loc. cit.*, p. 215.

[34] Eisenhart, *loc. cit.*, pp. 199–200.

any flat space. The existence of Einstein spaces of type (c) is proved here for the first time. The possibility of an E_4 of type (c) in a Euclidean S_5 was first indicated by Kasner.[35] He asked the question whether such E_4's actually existed. It follows from the above theorem or from (7.9) that the answer to this question is in the negative, that is,

THEOREM 7.2. *There are no Einstein spaces of dimensionality $N \geqq 4$ of negative mean curvature which are hypersurfaces of a Euclidean space.*

It is seen from (7.9) that there is an analogous theorem for spaces of constant negative curvature whose first fundamental form is positive definite. Another immediate result of Theorem 7.1 is

THEOREM 7.3. *Every Einstein space of dimensionality $N \geqq 4$ which is a proper hypersurface of a flat space is a hypersphere, hyperplane or developable hypersurface.*

It is clear that there are only a finite number[36] of E_N's of type (c) in any S_{N+1} (except for a motion in the S_{N+1}). All of these Einstein spaces have the same mean curvature according to (7.8). We note this fact in

THEOREM 7.4. *The mean curvature of a proper E_N which is not an S_N in an S_{N+1} of Riemann curvature K_0 is $(N-2)K_0$.*

The first fundamental form and the equations of the imbedding of an E_N of type (a) or type (b) in an S_{N+1} are known. According to Theorem 6.2 and Theorem 7.1, the corresponding results for type (c) are

THEOREM 7.5. *The first fundamental form of an E_N of type (c) may be written as*

$$ds^2 = \frac{e_1(dx^1)^2 + \cdots + e_P(dx^P)^2}{\left[1 + \frac{N-2}{P-1}\cdot\frac{K_0}{4}(e_1x^{1^2} + \cdots + e_Px^{P^2})\right]^2} + \frac{e_{P+1}(dx^{P+1})^2 + \cdots + e_N(dx^N)^2}{\left[1 + \frac{N-2}{N-P-1}\cdot\frac{K_0}{4}(e_{P+1}x^{P+1^2} + \cdots + e_Nx^{N^2})\right]^2},$$

where each e is $+1$ or -1. This E_N in an S_{N+1} is indeformable and is imbedded in the S_{N+1} by means of the algebraic equations

$$e_1z^{1^2} + \cdots + e_{P+1}z^{P+1^2} = \frac{P-1}{N-2}\cdot\frac{1}{K_0},$$

$$e_{P+2}z^{P+2^2} + \cdots + e_{N+2}z^{N+2^2} = \frac{N-P-1}{N-2}\cdot\frac{1}{K_0},$$

where the e's are $+1$ or -1 and the S_{N+1} is defined by

$$e_1z^{1^2} + \cdots + e_{N+2}z^{N+2^2} = \frac{1}{K_0}.$$

---

[35] E. Kasner, *Geometrical theorems on Einstein's cosmological equations*, American Journal of Mathematics, vol. 43 (1921), p. 219.

[36] The number of E_N's of type (c) is $\frac{1}{2}(N-2)$ or $\frac{1}{2}(N-3)$ according as N is even or odd.

Thus an E_N of type (c) is also the intersection of two spherical hypercylinders in a flat space of $N + 2$ dimensions. For $N = 4$, this situation was discussed by Kasner[37] who showed that the E_4 whose first fundamental form is

$$\frac{dx^{1^2} + dx^{2^2}}{\left[1 + \frac{K}{4}(x^{1^2} + x^{2^2})\right]^2} + \frac{dx^{3^2} + dx^{4^2}}{\left[1 + \frac{K}{4}(x^{3^2} + x^{4^2})\right]^2}$$

may be imbedded in a Euclidean S_6 by means of equations analogous to those of Theorem 7.5.

He also found that this space is the only E_4 whose first fundamental form is the sum of two forms, one involving only P of the variables, the other involving the remaining $4 - P$ variables. When the first fundamental form of a V_N may be written as

$$ds^2 = g_{pq}(x^r)dx^p dx^q + g_{ab}(x^c)dx^a dx^b, \tag{7.10}$$

the form is called *separable* and the two forms $g_{pq}dx^p dx^q$ and $g_{ab}dx^a dx^b$ are called its *components*. As shown by Bompiani, the necessary and sufficient condition that the subspaces x^p = constant, as well as the subspaces, x^a = constant, be totally geodesic in V_N is that the first fundamental form of V_N may be written as (7.10). It follows from Theorem 7.5 that the first fundamental form of an E_N of type (c) is separable and each component represents a space of constant curvature. However, as contrasted with Kasner's result, if $N > 4$ other E_N's than those of type (c) exist whose first fundamental form is separable. As this subject is not directly connected with the matter now being considered we simply state the result without proof.

THEOREM 7.6. *If the first fundamental form of an Einstein space of mean curvature κ is separable, each component is the first fundamental form of an Einstein space of mean curvature κ. If one component involves only one differential, $\kappa = 0$. Conversely, only the first fundamental forms of Einstein spaces are separable in this manner.*

By a repeated application of this theorem, we may obtain an obvious generalization which applies to an E_N whose first fundamental form is separable into more than two components.

8. Algebraic characterization of Einstein spaces E_N in an S_{N+1}

Closely related to the problem of finding the first fundamental form and the actual imbedding of an E_N in an S_{N+1} as considered in the preceding section is the algebraic characterization of such spaces. In a recent paper, Allendoerfer[38]

---

[37] E. Kasner, *An algebraic solution of the Einstein equations*, Transactions of the American Mathematical Society, vol. 27 (1925), pp. 103–104 and *Separable quadratic differential forms and Einstein equations*, Proceedings of the National Academy of Sciences, vol. 11 (1925), pp. 95–96.

[38] C. B. Allendoerfer, *Einstein spaces of class one*, Bulletin of the American Mathematical Society, vol. 43 (1937), pp. 265–270.

gave such a characterization of the E_N's with $\kappa \neq 0$ which may be imbedded in a flat space. He obtained two algebraic conditions which are satisfied by the Riemann curvature tensor of E_N if E_N is of class one. We now indicate how a similar characterization may be obtained for those E_N's which may be imbedded in an S_{N+1} of Riemann curvature K_0.

The characterization for an E_N of class one was obtained by purely algebraic methods from the Gauss equations for E_N and (7.2) which appear in Allendoerfer's paper as (6) and (1) respectively. The need for differentiating was avoided by using a theorem of Thomas[39] to show that the Codazzi equations of E_N in a flat space are consequences of the Gauss equations.

This theorem of Thomas states that under certain general conditions, the Codazzi equations for a hypersurface of a Euclidean space are consequences of the Gauss equations. It is easy to show, following the procedure of Thomas, that this theorem is true even when the enveloping space is not a Euclidean space but any space of constant curvature. (In the statement of this result in the Proceedings paper, the words "Riemann space" were incorrectly substituted for "space of constant curvature"). Hence in the generalization of Allendoerfer's work, it will suffice to follow his algebraic methods.

If the Gauss equations for E_N in a flat space are replaced by the Gauss equations (5.13) for E_N in an S_{N+1} of Riemann curvature K_0, Allendoerfer's work applies in all particulars with the understanding that R_{ijkl}, $g^{il}R_{ijkl}\left(\equiv \frac{R}{N} g_{jk}\right)$ and $g^{jk}R_{jk}(\equiv R)$ are replaced by $[R_{ijkl} - K_0(g_{ik}g_{jl} - g_{il}g_{jk})]$, $\left[\frac{R}{N} + (N-1)K_0\right]$ and $[R + N(N-1)K_0]$ respectively. Thus if these substitutions are made in the two conditions obtained by Allendoerfer, we obtain algebraic conditions that an E_N be imbedded in an S_{N+1} of Riemann curvature K_0. The restriction that $\kappa \neq 0$ is replaced by the assumption that $\kappa \neq (N-1)K_0$. However, it is an open question whether this algebraic characterization applies to any E_N's which were not discussed in §7, i.e., those E_N's for which the imbedding is not proper.

BROOKLYN COLLEGE,
BROOKLYN, N. Y.

[39] T. Y. Thomas, *Riemann spaces of class one and their characterization*, Acta Mathematica, vol. 67 (1936), p. 191.

ANNALS OF MATHEMATICS
Vol. 39, No. 4, October, 1938

NON-UNIQUE SOLUTIONS OF SYSTEMS OF FIRST ORDER ORDINARY DIFFERENTIAL EQUATIONS

BY O. H. HAMILTON[1]

(Received May 16, 1938; revised June 14, 1938)

1. Introduction

Work has been done in the study of non-unique solutions of systems of ordinary differential equations of the first order by Kamke, Osgood, Montel, Kneser, Fukuhara, Van Kampen, and others.[2] In general, these writers have considered differential equations in which the slope, dy_i/dx, is a continuous function of all the variables. Many of the results which these authors have obtained under the assumption of continuity have recently been obtained by H. J. Ettlinger (15) under weaker assumptions. This paper is an extension of Ettlinger's results.

The discussion will be concerned with a system of differential equations:

$$(A) \qquad y_i' = f_i(x, y_1, y_2, \cdots, y_n), \qquad i = 1, \cdots, n,$$

defined in a region, R, of an $n + 1$ dimensional Euclidean space. R is the region, $0 \leqq x \leqq 1$, $-\infty < y < \infty$. A set of functions, (B), $y_i = y_i(x)$, will be said to constitute a solution of the system, (A), through the point, P_0, $(x_0, y_1^0, y_2^0, \cdots, y_n^0)$, provided that for each integer i, $i = (1, n)$,

$$(B) \qquad y_i = y_i^0 + \int_{x_0}^{x} f_i[(t, y_1(t), \cdots, y_n(t)]\, dt.$$

This set of functions determines an arc y, having one endpoint in the set, $x = 0$, and the other in the set, $x = 1$, and which contains but one point in common with the set, $x = x_a$, $0 \leqq x_a \leqq 1$. Such an arc will be referred to as an integral curve of the system (A). If each function of a set (B) is an absolutely continuous function of x, then the arc y determined by the set will be said to be absolutely continuous; and if the family of all the functions in a collection of sets of functions such as (B) is equi-absolutely continuous, then the collection of curves determined by the various sets of functions in the collection will be said to be equi-absolutely continuous.

[1] The results contained in this paper were presented to the American Mathematical Society at its Christmas meeting in St. Louis in December, 1935. The writer wishes to acknowledge his indebtedness to Prof. H. J. Ettlinger for having interested him in the problems with which the paper is concerned, and for his criticism and help during the period of its preparation.

[2] See bibliography; a number following a name refers to the corresponding entry in the bibliography.

Ettlinger's assumptions are those indicated by 1), 2), and 3) below. We shall refer to these as Hypothesis (C).

1) $y_i' = f_i(x, y_1, y_2 \cdots, y_n)$, $i = 1, 2, 3, \cdots, n$, is a system (A) of n differential equations, each defined for all values of the variables in the region R, $0 \leqq x \leqq 1$; $-\infty < y < \infty$.

2) The Cauchy Field[3] in R determined by the given system of differential equations is an equi-absolutely continuous collection of curves for each point P_0 of R; and in any bounded subregion R_k of R, the collection of curves consisting of all the curves in the Cauchy Fields for all the points of the region R_k is an equi-absolutely continuous collection of curves.

3) For each integer i, y_i' is continuous in the variables $y_1, y_2, \cdots, y_n$ for each fixed x on the interval $(x_0, 1)$, except for a set of measure zero.

We shall refer to 1), 2), 3), and 4) as Hypothesis (C').

4) If P, $(x_p, y_1, y_2, \cdots, y_n)$, is any point of R and $g_i(x)$ is a system of equations determining an absolutely continuous curve containing P, then for each integer, r, $1 \leqq r \leqq n$, there exists a constant K_{Pr} such that $\lim_{x \to x_p} f_r[(x, g_1(x), g_2(x), \cdots, g_n(x)] = K_{Pr}$.

It should be noted that neither (C) nor (C') insures the continuity of y' in all the variables. From Ettlinger's conditions he obtained, among others, Properties (D) and (E) as stated below, and to which we shall refer by letter when necessary.

(D) If M is a bounded subset of R, the set of all integral curves of the system (A) which contain points of M is an equi-absolutely continuous collection of curves.

(E) A limit curve of a sequence of integral curves of the system (A) is an integral curve of the system.

Neither (C) nor (C') nor even continuity, as has been shown by the authors mentioned, insures that through a given point of R there is one and only one

---

[3] The Cauchy Field is defined as follows: Let P_0 be any point in R, and let T_{im} be a fine net on the interval, $(x_0, 1)$. The coördinates of P_0 are, $(x_0 y_1^0, y_2^0, \cdots, y_n^0)$. For each integer, m, the interval, $(x_0, 1)$ is divided into m subintervals by the points, $t_{0m}, t_{1m}, \cdots t_{nm}$, where x_0 is t_{0m}, and 1 is t_{mm}. For each fixed value of m, let y_m be a curve passing through P_0 and defined by the system, $y^i_{n_m}(x)$:

$$y^i_{n_m}(x) = y_i^0 + \int_{x_0}^{x} f_i[(t, y_1(x_0), y_2(x_0), \cdots, y_n(x_0)]\,dt,$$

on the interval, (x_0, t_{1m}), and in general,

$$y^i_{n_m}(x) = y^i_{n_m}(t_{j-1,m}) + \int_{t_{j-1,m}}^{x} f_i[(t, y^1_{n_m}(t_{j-1,m}), y^2_{n_m}(t_{j-1,m}), \cdots, y^n_{n_m}(t_{j-1,m})]\,dt$$

on the interval, $(t_{j-1,m}, t_{jm})$; $i = 1, 2, \cdots; j = 1, 2, \cdots m$. The set of curves thus defined through P_0 for all possible modes of division, both to the right and to the left from x_0 is called the Cauchy Field determined by the given system of differential equations and the point P_0.

integral curve of the system (A). Our results are concerned with the properties of collections of integral curves in R whose elements have points in common, and with the point sets associated with these collections.

2. General Theorems

DEFINITION: (See Kneser(4).) *If P_0 is a point of R and $B(P_0)$ is the set of points of R each of which is contained in some integral curve of the system* (A) *which contains P_0, then $B(P_0)$ is called the band solution of the system* (A) *through the point P_0.*

From property E it follows that the point set $B(P_0)$ is the sum of the elements of the equi-continuous collection of integral curves of the system (A) each of which passes through the point P_0.

(F) Kneser (4) first showed that the point set $x = x_a$, $0 \leqq x_a \leqq 1$ intersects $B(P_0)$ in a continuum. This property can be shown by methods similar to those used by Fukuhara (11) to follow from Hypothesis (C) and will be referred to as property (F).

In Theorems I–VI, inclusive, we assume Hypothesis (C).

THEOREM I: *If $P_1, P_2, \cdots$ is a sequence of points of R converging to a point P_0, then some subsequence of the band solutions $B(P_1), B(P_2), \cdots$ converges to a set of points containing P_0 which has property (F), and which is a subset of $B(P_0)$.*

PROOF. Some subsequence of the sequence of continua, $B(P_1)$, $B(P_2)$, $B(P_3), \cdots$ converges, by a theorem of Analysis Situs, to a sequential limiting set, $B'(P_0)$, which is a continuum. $B'(P_0)$ contains P_0, since $B(P_i)$ contains P_0 for each i. Let $B(P_{n1})$, $B(P_{n2}), \cdots$ be the subsequence of $B(P_1)$, $B(P_2)$, $B(P_3) \cdots$ converging to $B'(P_0)$, and let the common part of the set $x = x_a$, $0 \leqq x_a \leqq 1$, and the point set $B(P_{ni})$ be designated by F_{ni}. By Kneser's theorem, F_{ni} is a continuum for each integer, i. The sequence of continua, $F_{n1}, F_{n2}, \cdots$ converges to the continuum, F', which is the common part of $B'(P_0)$ and the point set $x = x_a$. Thus $B'(P_0)$ has property (F). Every point Q of $B'(P_0)$ is a limit point of a sequence of points, $Q_{m1}, Q_{m2}, \cdots$ where, for each integer i, Q_{mi} lies on an integral curve y_{mi} through P_{mi}. This sequence of integral curves contains a subsequence which, by property (E), converges to an integral curve through P_0 and which contains Q. Q is therefore a point of $B(P_0)$, and $B'(P_0)$ is a subset of $B(P_0)$.

THEOREM II: *If M is a connected subset of R and $B(M)$ is the set of points, each of which is contained in an integral curve of* (A) *which contains a point of M, then the common part of the point set $x = x_a$, $0 \leqq x_a \leqq 1$, and $B(M)$ is connected.*

PROOF. Let N be the common part of the point sets $x = x_a$ and $B(M)$. Suppose that N is the sum of two point sets, N_1 and N_2, neither of which contains a point or a limit point of the other. Let M_1 be the subset of M consisting of all the points of M which lie on some integral curve of (A) which contains a point of N_1 and let M_2 be the subset of M consisting of all the points of M which lie on some integral curve of (A) which contains a point of N_2. M_1 and M_2 are mutually exclusive. For suppose that M_1 and M_2 have a point P in com-

mon. Let N_P be the common part of the sets $B(P)$ and $x = x_a$, $0 \leqq x_a \leqq 1$. By property (F), N_P is a continuum. N_P must then be a subset of N_1 or of N_2. But since P belongs to both M_1 and M_2, N_P must contain a point of both N_1 and N_2. It follows that M_1 and M_2 are mutually exclusive. Since M is connected, either M_1 contains a limit point of M_2 or M_2 contains a limit point of M_1. Suppose that M_2 contains a limit point Q of M_1. Let N_Q be the common part of $B(Q)$ and the set, $x = x_a$. N_Q is a continuum by property (F), and is therefore a subset of either N_1 or N_2. Since Q belongs to M_2, N_Q is a subset of N_2. Let $Q_1, Q_2, \cdots$ be a sequence of points of M_1 converging to Q, and let y_i, for each integer i, be an integral curve of (A) through Q_i. By property (E) and Ascoli's theorem on the limit functions of equi-continuous collections of functions, some sub-sequence, $y_{n1}, y_{n2}, \cdots$ of the sequence, $y_1, y_2, \cdots$ converges to an integral curve y_Q through Q. Let B_{ni}, for each integer i be the point which y_{ni} has in common with N_1. The sequence of points $B_{n1}, B_{n2}, \cdots$ converges to a point of N_Q and therefore to a point of N_2. This contradicts the assumption that N_2 does not contain a limit point of N_1. We arrive at a similar contradiction if we assume that M_1 contains a limit point of M_2. It follows that N is a continuum.

THEOREM III: *If* (1) *the system* (A) *has more than one equation*; (2) (A) *has unique solutions to the* {*right* / *left*} *from each point of* R; (3) P_0 *is a point of* R; (4) $B(P_0)$ *is the band solution generated by* P_0; *and* (5) N *is the common part of the point sets,* $B(P_0)$ *and* $x = x_a$, $0 \leqq x_a \leqq 1$; *then* N *does not separate the space* $x = x_a$.

PROOF: Suppose (A) has unique solutions to the left from each point of R and that N separates the space $x = x_a$. There exist two points A and B, each belonging to the set $x = x_a$, which do not belong to N and which are separated from each other by N. Let M be the connected set consisting of all the points of the set $x = x_0$, except P_0. M is a connected set of points. $B(M)$ and the common part W_M of $B(M)$ and the set $x = x_a$ are connected sets by Theorem II. But W_M contains the points A and B and therefore contains a point Q of N. There is therefore an integral curve to the left from Q to P_0, since Q is in N, a subset of $B(P_0)$; and there is also another integral curve to the left from Q to a point of M, since Q is in $B(M)$. This contradicts the assumption that the solution of (A) to the left from each point of R is unique. The assumption that N separates the space $x = x_a$ thus leads to a contradiction. We arrive at a similar contradiction if the system (A) has unique solutions to the right from each point of R.

It should be noted that if the system (A) contains only one equation, $y' = f(x, y)$, the common part of the line $x = x_a$ and $B(P_0)$ is a straight line interval or a single point, and that its boundary in the set $x = x_a$ is a pair of points or a single point.

DEFINITION. *The diameter of a band solution. If* (A) *has a band solution* $B(P_0)$ *through a point* P_0 *of* R, *let* d_a *be the diameter of the set which is the common part of* $B(P_0)$ *and the set* $x = x_a$, $0 \leqq x_a \leqq 1$. *The least upper bound* d *of the*

numbers d_a for all x_a's will be called the diameter of the band solution, $B(P_0)$. The *diameter of a subset of a band solution* will be defined in a similar manner.

THEOREM IV: *If c is a real positive number, then M_c, the set of points in R which generate band solutions of diameter greater than or equal to c, is a closed set of points.*

PROOF. Let $P_1, P_2, P_3, \cdots$ be a sequence of points of R converging to a point P, such that for each integer i, P_i generates a band solution of the system (A) of diameter greater than or equal to c. For each i let N_i be the common part of a set $x = x_i$, $0 \leqq x_i \leqq 1$, and $B(P_i)$ which is of diameter greater than or equal to c. N_i exists for each integer i since $B(P_i)$ contains a set of diameter greater than or equal to c. Some subsequence of the sequence of continua, $N_1, N_2, N_3, \cdots$ converges to a sequential limiting set N_0 which is of diameter greater than or equal to c, and which lies in a set $x = x_a$, $0 \leqq x_a, \leqq 1$. Furthermore, if Q is any point of N_0, there exists an integral curve of the system (A) through the points P and Q as can be shown as follows. For each integer i let y_i be an integral curve through P_i and a point Q_i of N_i such that the sequence of points Q_i converges to Q. Some subsequence of the sequence of integral curves, $y_1, y_2, y_3, \cdots$ converges to an integral curve of the system (A), by property E, which goes through both P and Q. N_0 is therefore a subset of the band solution of (A) through P of diameter greater than or equal to c.

THEOREM V: *The set of points in R which generate non-unique solutions of the system* (A) *is the sum of a countable number of closed point sets.*

PROOF: For each integer i let M_i be the set of points in R each of which generates a band solution of diameter greater than or equal to $1/i$. By Theorem IV, it follows that M_i is closed for each integer i. Furthermore, each point of R which generates a non-unique solution of the system (A) belongs to one of these sets. Since the sequence $M_1, M_2, M_3, \cdots$ is countable, the theorem is true.

COROLLARY: *The set of points in R which generate unique solutions of the system* (A) *is an inner limiting set.*

PROOF: This follows immediately from the fact that the complement of the sum of a countable number of closed point sets in a domain is an inner limiting set.

In the Theorems which have gone before we have assumed only that the system (A) of differential equations satisfies Hypothesis (C). In what follows we shall assume that (C′) holds and that $n = 1$.

3. Theorems Which Hold for the Special Case $n = 1$

THEOREM VI: *If Hypothesis* (C′) *holds for the differential equation, $y' = f(x, y)$, and P_0 is a point of R, then there do not exist uncountably many real numbers e, such that the differential equation $y' = f(x, y) + e$ has a non-unique solution through P_0.*

PROOF: Suppose the theorem is not true and that there exists an uncountable collection G, of numbers e and a point P_0 of R such that there is a non-unique solution of the differential equation $y' = f(x, y) + e$ through P_0 for each e in G.

Kamke[13] and others have shown that under such circumstances, if e_1 and e_2 are two numbers of the collection G, then no two integral curves y_1 and y_2 of the equations $y' = f(x, y) + e_1$ and $y' = f(x, y) + e_2$ which go through P_0 can intersect at any other point. For each e of the collection G, there is a two dimensional domain D_e contained between the maximum and minimum curves of the band solution of $y' = f(x, y) + e$ which goes through P_0. These domains must be mutually exclusive. Thus we have existing in the XY plane an uncountable collection of mutually exclusive domains. This is a contradiction of the Lindelöf [2] Covering Theorem. Theorem VI is therefore true.

THEOREM VII: *If $y' = f(x, y)$ is a differential equation satisfying Hypothesis (C'), if K is any countable set of points in R and d is any positive number, there exists an uncountable collection G of real numbers e, such that $|e| < d$ and such that the differential equation $y' = f(x, y) + e$ has a unique solution through each point of the countable set K for each e belonging to G.*

(The statement of this theorem is equivalent to the statement that any differential equation satisfying Hypothesis (C') can be approximated as close as one pleases by a differential equation which has a unique solution through each of a given countable set of points in R. That countable set may, of course, be everywhere dense in R.)

PROOF. Let d be any positive number. There exists an uncountable collection G of numbers each less in absolute value than d. Let $P_1, P_2, P_3, \cdots$ designate a simple sequence of the points of the countable set K of points in R. For each integer j, let B_j designate the countable subcollection of G such that if e is a number of the subcollection B_j, then the solution of the differential equation $y' = f(x, y) + e$ through the point P_j is non-unique. B_j is countable by Theorem VI. Let $B = \sum_1^\infty B_j$. Then the collection B is countable, the collection $G - B$, is uncountable; and for each number e, in $G - B$, $y' = f(x, y) + e$ has a unique solution through each point of the set K.

THEOREM VIII: *A necessary and sufficient condition that there exists an everywhere dense set of points M in R such that $y' = f(x, y)$ has a unique solution through each point of M is that $y' = f(x, y)$ has a unique solution either to the right or to the left from each point of R.*

PROOF: Van Kampen (16) has shown that under these conditions the set of points on any line $x = a$ in R which generate non-unique solutions of $y' = f(x, y)$ is countable. It follows that the set of points which generate unique solutions is everywhere dense on each such line $x = a$, and is therefore everywhere dense in K. The condition is therefore sufficient.

That the condition is necessary may be shown as follows. Suppose that there exists an everywhere dense set of points M in R such that $y' = f(x, y)$ has a unique solution through each point of M and that from some point P of R there is a non-unique solution both to the right and to the left. There is a point Q of M lying in $B(P)$ and which is to the right of P. There is an integral curve of $y' = f(x, y)$ which contains both Q and P. Since there is a non-unique solution of $y' = f(x, y)$ to the left from P there is a non-unique solution to the left

from Q. This contradicts the fact that Q belongs to M. The given condition is therefore necessary.

THEOREM IX: *Using the notation and hypothesis of Theorem VII, there exists an uncountable collection G of real numbers, all less in absolute value than a previously chosen real number d, and such that*; 1) *If P is any point of R and e is any number of the collection G, then the differential equation $y' = f(x, y) + e$ has a unique solution in one direction only from P, and,* 2) *for each fixed number $\bar{e}$ of G, the set of points of R which generate non-unique band solutions of the system $y' = f(x, y) + \bar{e}$ lies on a point set which is the sum of a countable collection of arcs and which is therefore of two dimensional measure zero.*

PROOF: Let d be any positive number. By Theorem VII there exists an uncountable collection G of real numbers e, all less in absolute value than d, and such that the differential equation $y' = f(x, y)$ has a unique solution through each point of R which has rational coördinates.

By Theorem VIII, for each e belonging to G, $y' = f(x, y) + e$ has a unique solution either to the right or to the left. The first statement in the theorem is therefore true.

Let e be any number of the collection G. For each point P having rational coördinates let E_P be the unique integral curve through P of the differential equation $y' = f(x, y) + e$. The set H of such curves is countable. Let T be any point of R which generates a non-unique solution $B(T)$ of $y' = f(x, y) + e$. There exists a point P_1 having rational coördinates and lying in $B(T)$. P_1 lies on an integral curve h of the collection H. h obviously contains the point T. Since T is any point of R which generates a non-unique solution of $y' = f(x, y)$, the set of all such points lies on H^*, the set of points lying on the collection of curves H. H^* is therefore of two dimensional measure zero.

4. An Example

We now give an example of a differential equation, $y' = f(x, y)$, which satisfies Hypothesis (C), but not (C'), 4), and for which Theorem VI does not hold, and consequently, for which Theorems VII, VIII, and IX do not necessarily hold.

$$\begin{aligned}
\text{Let } y' &= 0 && \text{for} && x = 0, && -\infty < y < \infty \\
&= y/x && \text{for} && 0 < x \leqq 1, && -2x \leqq y \leqq 2x \\
&= 2 && \text{for} && 0 < x \leqq 1, && 2x \leqq y < \infty \\
&= -2 && \text{for} && 0 < x \leqq 1, && -\infty < y \leqq -2x
\end{aligned}$$

It is clear that part 4) of Hypothesis (C') is not satisfied at (0, 0) for this differential equation. Consider the band solution of this differential equation through (0, 0). $y = 2x$ and $y = -2x$ are the maximum and minimum solutions of the band solution. For $y' = y/x + K$, where $0 < K < 1$, $y = (2 + K)x$ and $y = (K - 2)x$ are the maximum and minimum solutions through (0, 0). It is clear that the band solution through (0, 0) for $y' = y/x + K$ has points

other than (0, 0) in common with the band solution through (0, 0) of $y' = y/x$, and Theorem VI does not hold.

THE UNIVERSITY OF TEXAS,
AUSTIN, TEXAS.

BIBLIOGRAPHY

(1) VITALI, G. *Sull Integrazione per serie.* Palermo Rend. (23), p. 139.

(2) LINDELÖF, E. *Remarques sur un théorème fondamental de la théorie des ensembles.* Acta. Math., Vol. 29 (1905), p. 188.

(3) PERRON, O. *Ein neuer Existenzbeweis für die Integrale der Differentialgleichung, $y' = f(x, y)$.* Math. Ann., Vol. 76 (1915), p. 471.

(4) KNESER, H. *Über die Lösungen eines Systems gewöhnlicher Differentialgleichungen das der Lipschitzschen Bedingung nicht genügt.* Sitz. Ber. Akad. der Wissn.; phys. math. Klasse (1923), pp. 205–217.

(5) MONTEL, P. *Sur l'intégrale supérieure et l'intégrale inférieure d'une equation différentielle.* Bull. Sc. Math., Vol. 50 (1926), pp. 205–217.

(6) ETTLINGER, H. J. *On Multiple Iterated Integrals.* Am. Journal Math., Vol. 48 (1926), pp. 215–216.

(7) MULLER, M. *Beweis eines Satzes des Herrn. H. Kneser.* Math. Zeit., Vol. 28 (1928), pp. 349–355.

(8) FUKUHARA, MASUO. *Sur le théorème d'existence des intégrales des équationes différentiales du premier ordre.* Jap. Journal Math., Vol. 5 (1928), pp. 345–350.

(9) FUKUHARA, MASUO *Sur les systèmes des équations différentielles ordinaires.* Proc. Imperial. Acad. of Japan, Vol. 4 (1928), p. 448.

(10) FUKUHARA, MASUO. *Sur les systèmes des équations différentielles ordinaires.* Jap. Journal of Math., Vol. 6, pp. 269–299; Vol. 7, pp. 173–186.

(11) FUKUHARA, MASUO. *Sur l'ensemble des courbes intégrales d'un systeme d'équations différentielles ordinaires.* Proc. Imper. Acad. of Japan., Vol. 6 (1930), pp. 360–362.

(12) KAMKE, E. *Zur Theorie der Systeme gewöhnlicher Differentialgleichungen.* Jour. reine und angew. Math., Vol. 16, p. 194.

(13) KAMKE, E. *Zur Theorie der Systeme gewöhnlicher Differentialgleichungen.* Acta. Math., Vol. 58 (1932), pp. 57–85.

(14) DIGEL, E. *Zur einem Beispel von Nagumo und Fukuhara.* Math. Zeit., Vol. 39 (1934), pp. 157–160.

(15) ETTLINGER, H. J. *Cauchy Fields of ordinary differential equations.* Abstract, Bull. Amer. Math. Soc., Vol. XLII, Jan. 1936.

(16) VAN KAMPEN, E. R. *Remarks on systems of Ordinary Differential Equations.* Am. Journal Math., Vol. 59, pp. 144–152.

ANNALS OF MATHEMATICS
Vol. 39, No. 4, October, 1938

AN ABSTRACT FORM OF THE COVERING THEOREMS OF TOPOLOGY

BY KARL MENGER

(Received January 27, 1938)

The general rôle of the covering theorems in topology is this: Given 1) a subset A' of a space A, and considering 2) the set B of all subsets of A and 3) the relation between points of A and subsets of A expressed by the words "a point of A is an interior point of a subset of A," then the covering theorems state the equivalence of a condensation property and a covering property of A'. More specifically: By calling a set B' whose elements are subsets of A (thus a subset B' of B)a *covering system of the subset* A' of A if each point of A' is an interior point of at least one subset of A that is an element of B', then two of the most important covering theorems may be formulated in the following way: The theorem for separable sets states the equivalence of the covering property "each covering system of each subset A'' of A' contains a denumerable covering system of A''" and the condensation property "each non-denumerable subset A'' of A' contains at least one point p such that each set which contains p in its interior contains a non-denumerable subset of A' in its interior." The theorem for compact sets in separable spaces states the equivalence of the covering property "each covering system of A' contains a finite covering system of A'" and the condensation property "for each infinite subset A'' of A' the set A' contains at least one point p such that each set which contains p in its interior contains an infinite subset of A'' in its interior."

It is the purpose of the present paper to show that, in the last analysis, it is not essential to view the covering theorems in this particular rôle as dealing with a space or a subset of a space, with a covering system of the set, and with the extraction of something of a smaller power. In fact, we are going to prove theorems that hold for an abstract set (not only for a subset of a space) and for a general mapping of this first abstract set on another abstract set (not necessarily a set consisting of subsets of the first set, and not necessarily a mapping by means of the relation "interior point of a set"). A little less obvious than the possibility of generalizing the covering theorems in these three directions[1] is the fact that their quantitative character can also be eliminated to a large extent. The quantitative covering properties stating the possibility of extracting a covering of a smaller power may be replaced by properties stating that the considered mapping of A' on B' contains a "submapping" on a subset of B' belonging to a prescribed system $\mathfrak{B}^*$ of subsets of B. The quantitative condensation properties stating the existence of cluster-points for sets of high power

[1] These were indicated by the author in *Ergebnisse eines mathematischen Kolloquiums* 2, 1930, p. 26.

may be replaced by properties stating the existence of certain points associated with sets that do not belong to a prescribed system $\mathfrak{A}^*$ of subsets of A. The prescribed systems of subsets of A and B may, but need not, consist of all sets whose power is small.

The method of proof of this detopologization of the covering theorems consists in the application of the logic of relations to an analysis of the proof of the covering theorems due to Kuratowski and Sierpinski.[2]

A relation between the elements of two sets A and B will be considered as a set of ordered pairs consisting of an element of A and an element of B. Instead of speaking of such a relation we shall deal with a subset R of the set $A \times B$ consisting of the ordered pairs (a, b) of an element a of A and an element b of B. Instead of saying that the elements a and b are in the relation considered we shall say that the pair (a, b) is an element of the set R.

It will be clear from the content of the present paper that the abstract method of formulating and proving the covering theorems may be applied to other parts of the geometry of point sets. Some problems will be indicated at the end.

We assume that five sets are given: 1) an abstract set A, 2) an abstract set B, 3) a set $R \subseteq A \times B$, i.e. a set of ordered pairs (a, b) where $a \in A$ (a is element of A) and $b \in B$, 4) a set $\mathfrak{A}^*$ whose elements are subsets of A, 5) a set $\mathfrak{B}^*$ whose elements are subsets of B.

DEFINITION 1: *If $A' \subseteq A$ and $b \in B$ we denote by $A'(b)$ the set of all a' such that $a' \in A'$ and $(a', b) \in R$.*

DEFINITION 2: *We denote by $\mathfrak{R}$ the set of all ordered pairs (A', B') such that $A' \subseteq A$, $B' \subseteq B$ and $A' = \sum_{b' \in B'} A'(b')$.*

REMARK 2′: If $(A', B') \in \mathfrak{R}$ and $A'' \subseteq A'$, then $(A'', B') \in \mathfrak{R}$.

REMARK 2″: If A' is vacuous, then $(A', B') \in \mathfrak{R}$ for each $B' \subseteq B$.

DEFINITION 3: *If $A'' \subseteq A$ we denote by A''_* the set of all $a \in A$ such that for each b for which $(a, b) \in R$ it follows that $A''(b)$ non $\in \mathfrak{A}^*$.*

DEFINITION 4: *If $A'' \subseteq A' \subseteq A$ and $A' \cdot A''_*$ is not vacuous, then we shall say that A'' is $\mathfrak{A}^*$-condensed in A'.*

The hypotheses of the covering theorems are the following special cases of the previously mentioned assumptions: A is a topological space, B is the set of all subsets of A, the set R contains those and only those pairs (a, b) consisting of a point a of A and an element b of B (i.e. a subset b of A) for which a is interior point of b. Consequently, in this case, a pair (A', B') belongs to $\mathfrak{R}$ if and only if each point of A' is interior to at least one subset of A that is an element of B', or, as it is usually expressed, if and only if B' is a covering system of A'. If in this case for a certain ordinal number $\mathfrak{n}$, the set $\mathfrak{A}^*$ consists of those and only those subsets of A whose power is $\leqq \mathfrak{n}$, then A''_* is the set of all those points a of A which have the following property: Each set b which contains a in its interior also contains in its interior each point of a subset of A'' whose

[2] Fundamenta Mathematicae 2, 1921, p. 172.

power is $> \mathfrak{n}$. If, in particular, $\mathfrak{n} = \aleph_0$, then the set A''_* consists of all points of condensation of A''. In this case A'' is $\mathfrak{A}^*$-condensed in A' if A' contains at least one point of condensation of A''. If $\mathfrak{A}^*$ consists of all subsets of A whose power is $< \mathfrak{n}$, then A''_* is the set of all those points a of A which have the following property: Each set b which contains a in its interior also contains in its interior each point of a subset of A'' with a power $\geqq \mathfrak{n}$. If, in particular, $\mathfrak{n} = \aleph_0$, then A''_* consists of the cluster-points of A''. In this case, a set A'' is $\mathfrak{A}^*$-condensed in A', if A'' contains at least one cluster-point of A'.

The mapping property and the first generalized condensation property of a set A' which will subsequently be related are given by the following two definitions:

DEFINITION 5: *A' has the property M if for each B' the relation*

1) $$(A', B') \in \mathfrak{R}$$

implies the existence of B^* such that

2) $B^* \subseteq B'$, 3) $(A', B^*) \in \mathfrak{R}$ 4) $B^* \in \mathfrak{B}^*$.

DEFINITION 6: *A' has the property C I if for each A'' the relations*

5) $A'' \subseteq A'$ *and* 6) A'' *non* $\in \mathfrak{A}^*$

imply that A'' is $\mathfrak{A}^*$-condensed in A', i.e. that

7) $$A' \cdot A''_* \text{ is not vacuous.}$$

We shall now formulate a hypothesis which will enable us to accomplish the first half of our task: the derivation of the generalized condensation properties from the generalized covering properties.

HYPOTHESIS H_1: *If $B^* \in \mathfrak{B}^*$ and if B^* is mapped on a subset of $\mathfrak{A}^*$, i.e., if with each element b of B^* an element of A^*_b of $\mathfrak{A}^*$ is associated, then $\sum_{b \in B^*} A_b \in \mathfrak{A}^*$.*

THEOREM I_1: *Hypothesis H_1 implies: If a set A' has the property M, then A' has the property C I.*

We prove this theorem by showing that from hypothesis H_1 it follows: For each set A' with the property M and each A'' the conditions 5) and non-7) imply the condition non-6).

Let us, in fact, assume that hypothesis H_1 holds, that A' has the property M, and that conditions 5) and non-7) are satisfied. Condition non-7) (i.e., that $A' \cdot A''_*$ is vacuous) may be formulated in the following way: $a \in A'$ implies a *non* $\in A''_*$ for each a. By virtue of definition 3 from the last assumption it follows:

8) For each $a \in A'$ there exists a b_a such that $(a, b_a) \in R$ and $A''(b_a) \in \mathfrak{A}^*$. We denote by B' the set of all b satisfying the condition

9) $$A''(b) \in \mathfrak{A}^*.$$

By virtue of the definitions 1 and 2, condition 8) implies

1) $$(A', B') \in \mathfrak{R}.$$

Since A' has the property M there exists a B^* such that conditions 2), 3), 4) are satisfied. Condition 3) and assumption 5) imply $(A'', B^*) \in \mathfrak{R}$ by virtue of the remark 2'. According to definition 2 we have thus $A'' = \sum_{b^* \in B^*} A''(b^*)$. This formula, on account of conditions 9) and 4) and by virtue of hypothesis H_1, implies $A'' \in \mathfrak{A}^*$, i.e. non-6). This completes the proof of theorem I_1.

We shall now formulate a stronger condensation property of a set A', namely, that each set A'' which is $\subseteq A'$ and does not belong to $\mathfrak{A}^*$ is $\mathfrak{A}^*$-condensed in itself (in A'', not only in A').

DEFINITION 7: *A' has the property of C II if for each A'' the relations*

5) $A'' \subseteq A'$ *and* 6) A'' *non* $\in \mathfrak{A}^*$

imply that A'' is $\mathfrak{A}^*$-condensed in A'', i.e. that

7') $$A'' \cdot A''_* \text{ is not vacuous.}$$

REMARK 3: If A' has the property C II, then A' has the property C I. In fact, since $A'' \subseteq A'$ implies that $A'' \cdot A''_* \subseteq A' \cdot A''_*$, it follows that 7') implies 7).

REMARK 4: If A' has the property C II, then each subset of A' has the property C II. Let A' be a set with the property C II, and $A'' \subseteq A'$. Then for each A''' which satisfies the conditions $A''' \subseteq A''$ and A''' *non* $\in \mathfrak{A}^*$ we also have $A''' \subseteq A'$. Thus by virtue of the property C II of A' the set $A''' \cdot A'''_*$ is not vacuous.

THEOREM II_1: *Hypothesis H_1 implies: If each subset of a set A' has the property M, then A' has the property C II.*

Let us assume that hypothesis H_1 holds and that each subset of A' has the property M. Let then A'' be any set satisfying conditions 5) and 6). In order to prove theorem II_1 we have to show that 7' holds. From 5) it follows that A'' has the property M. By virtue of theorem I_1 the set A'' has therefore the property C I, that is to say, for each set $A''' \subseteq A''$ such that A''' *non* $\in \mathfrak{A}^*$ the set $A'' \cdot A'''_*$ is not vacuous. In particular, we see on account of 6) that for A'' itself the set $A'' \cdot A''_*$ is not vacuous. Condition 7') is thus satisfied, and this completes the proof of theorem II_1.

Before solving the second half of our problem we shall apply the theorems I_1 and II_1 to some covering problems in the ordinary sense where A is a space, B the set of all subsets of A, and R consists of those pairs formed by a point a of A and a subset b of A for which a is interior to b. In this case a pair (A', B') belongs to $\mathfrak{R}$ if and only if B' is a covering system of A'. Let $\mathfrak{n}$ be an infinite cardinal number.

EXAMPLE 1: Let $\mathfrak{A}^*$ be the set of all subsets of A whose power is $\leqq \mathfrak{n}$, and $\mathfrak{B}^*$ the set of all subsets of B whose power is $\leqq \mathfrak{n}$, i.e. the system of all those sets of subsets of A which contain $\mathfrak{n}$ or less subsets of A. Then hypothesis H_1 is

satisfied. For if B^* is a set of a power $\leqq \mathfrak{n}$ and with each element b of B^* a set A_b^* of a power $\leqq \mathfrak{n}$ is associated, then $\sum_{b \in B^*} A_b$ is a set of a power $\leqq \mathfrak{n}$. Hence according to theorem I_1 each set with the property M has the property C I. The set A', in this case, has the property M if each covering system B' of A' contains a covering system B^* of A' which consists of at most $\mathfrak{n}$ subsets of A. The set A' has the property C I if for each subset A'' of A' which has a power $> \mathfrak{n}$ there exists at least one point a in A' such that each set which contains a in its interior, contains in its interior a subset of A' whose power is $> \mathfrak{n}$.

EXAMPLE 2: Hypothesis I does not hold if $\mathfrak{A}^*$ is the set of all subsets of A whose power is $< \mathfrak{n}$ and $\mathfrak{B}^*$ is the set of all subsets of B whose power is $\leqq \mathfrak{n}$.

EXAMPLE 3: Hypothesis H_1 holds if $\mathfrak{A}^*$ is the set of all subsets of A whose power is $\leqq \mathfrak{n}$ and $\mathfrak{B}^*$ is the set of all subsets of B whose power is $< \mathfrak{n}$. But the application of theorem I_1 in this case merely yields a special case of the result obtained in example 1.

EXAMPLE 4: Hypothesis H_1 holds if $\mathfrak{A}^*$ is the set of all subsets of A whose power is $< \mathfrak{n}$ and $\mathfrak{B}^*$ the set of all subsets of B whose power is $< \mathfrak{n}$.

EXAMPLE 5: If $\mathfrak{n}$ is a finite cardinal number > 0 and $\mathfrak{A}^*$ the system of all sets of a power $\leqq \mathfrak{n}$, then hypothesis H_1 is satisfied if and only if $\mathfrak{B}^*$ is the system of all sets consisting of at most one subset of A. If $\mathfrak{B}^*$ is the system of all subsets of B which contain exactly one element, then hypothesis H_1 is satisfied for each system $\mathfrak{A}^*$. The result of the theorem in these cases is trivial.

EXAMPLE 6: Hypothesis H_1 is satisfied if $\mathfrak{B}^*$ consists of the denumerable sets and $\mathfrak{A}^*$ is the set of all subsets of A that are either of first category in A (i.e. a sum of a denumerable set of sets that are nowhere dense in A) or an F_σ in A (i.e. a sum of a denumerable set of sets that are closed in A) or an F_σ in A of a dimension $\leqq n$ in the sense of the theory of dimension.

The solution of the second half of our task, the derivation of the generalized covering properties from the generalized condensation properties will be based on a lemma in which use is made of the following definition.

DEFINITION 7: *If $S = \{s_1, s_2, \cdots, s_\xi, \cdots\}$ is a well ordered set, then for each $s_\xi \in S$ we shall denote by S_ξ the well ordered set of all $s_\eta \in S$ for which $\eta < \xi$.*

LEMMA: *If A' and B' are two given sets such that $(A', B') \in \mathfrak{R}$, then there exist two sets A'' and B'' satisfying the condition*

1) $$A'' \subseteq A', \qquad B'' \subseteq B', \qquad (A', B'') \in \mathfrak{R}$$

and such that A'' and B'' may be well ordered

$$A'' = \{a_1, a_2, \cdots, a_\xi, \cdots\}, \qquad B'' = \{b_1, b_2, \cdots, b_\xi, \cdots\}$$

according to the following conditions:

2) $$(a_\xi, b_\xi) \in R \text{ for each } \xi,$$

3) $A''(b_\xi) \subseteq A_\xi'' + \{a_\xi\}$ *for each ξ where $A_\xi'' + \{a_\xi\}$, according to definition 7, denotes the well ordered set of all a_η with $\eta \leqq \xi$,*

4) *for each* $b_\xi \in B''$ *we have* (A', B''_ξ) *non* $\in \mathfrak{R}$.

If B' *is given as a well ordered set*

(I) $$B' = \{b'_1, b'_2, \cdots, b'_\xi, \cdots\},$$

then the well ordered set B'' *may be chosen so as to satisfy in addition the condition*

5) *If* $b_\xi = b'_{\xi'}$, *then* $\xi \leqq \xi'$ *for each* ξ.

We consider the given set A' and the set B' in the well ordered form (I). If A' is vacuous, then according to remark 2″ we satisfy the conditions 1)–5) by choosing A'' and B'' vacuous. We thus assume that A' is not vacuous. From $(A', B') \in \mathfrak{R}$ and definition 2 it follows that there exists at least one element b' of B', and at least one element a' of A' such that $(a', b') \in R$. We denote by b_1 the first element in (I) for which the set $A'(b_1)$ is not vacuous, and we call a_1 one of the elements of this set. Now we make the following inductive assumption for an ordinal number ξ_0 : For each ordinal number $\xi < \xi_0$ an element a_ξ of A' and an element $b_\xi = b'_{\xi'}$ of B' are determined in such a way that the following conditions hold:

$2_{\xi<\xi_0}$) $(a_\xi, b_\xi) \in R$ for each $\xi < \xi_0$.

$3_{\xi<\xi_0}$) If $\eta < \xi_0$, $\xi < \xi_0$, and $(a_\eta, b_\xi) \in R$, then $\eta \leqq \xi$.

$4_{\xi<\xi_0}$) For each $\xi < \xi_0$ the set $A' - \sum_{\eta<\xi} A'(b_\eta)$ is not vacuous.

$5_{\xi<\xi_0}$) If $\eta = \xi$ and $b_\eta = b'_{\eta'}$, then $\eta < \eta'$.

Then we distinguish two cases:

First case, in which we have

(II) $$A' \subseteq \sum_{\xi<\xi_0} A'(b_\xi).$$

In this case we call A'' the set of all a_ξ for $\xi < \xi_0$, B'' the set of all b_ξ for $\xi < \xi_0$. Then we have $A'' \subseteq A'$, $B'' \subseteq B'$, and $(A', B'') \in \mathfrak{R}$ on account of (II), and thus satisfy condition 1). Besides, in this case the conditions 2)–5) are satisfied by virtue of $2_{\xi<\xi_0}$)–$5_{\xi<\xi_0}$).

Second case, in which (II) does not hold and, consequently, the set $A' - \sum_{\xi<\xi_0} A'(b_\xi)$ which may be denoted by $A'(-\xi_0)$ is not vacuous. In this case from $(A', B') \in \mathfrak{R}$ and definition 3 it follows that there exists at least one element $b'_{\xi'}$ of B' and at least one element $a \in A'(-\xi_0)$ such that $(a, b'_{\xi'}) \in R$. We call b_{ξ_0} the first element $b'_{\xi'}$ in (I) for which the set $A'(-\xi_0) \cdot A'(b'_{\xi'})$ is not vacuous, and we call a_{ξ_0} one of the elements of this set. Then we have $a_{\xi_0} \in A'(-\xi_0)$ and $(a_{\xi_0}, b_{\xi_0}) \in R$, and thus satisfy condition $2_{\xi\leqq\xi_0}$). From $5_{\xi<\xi_0}$) and the determination of ξ_0 it follows that condition $5_{\xi\leqq\xi_0}$) is satisfied. So is condition $3_{\xi\leqq\xi_0}$): If $\eta \leqq \xi_0$, $\xi \leqq \xi_0$ and $(a_\eta, b_\xi) \in R$, then $\eta \leqq \xi$. For if $\eta < \xi_0$, $\xi \leqq \xi_0$ and $(a_\eta, b_\xi) \in R$, then $\eta \leqq \xi$ by virtue of $3_{\xi<\xi_0}$). If $\eta = \xi_0$, $\xi \leqq \xi_0$, and

$(a_{\xi_0}, b_\xi) \in R$, then $\xi = \xi_0$, thus $\eta = \xi$ on account of the way that a_{ξ_0} was chosen. Finally condition $4_{\xi<\xi_0}$) and the assumption that $A'(-\xi_0)$ is not vacuous imply condition $4_{\xi\leqq\xi_0}$). We see thus: If the elements $b_1, \cdots, b_\xi, \cdots$ and $a_1, \cdots, a_\xi, \cdots$ are determined for each $\xi < \xi_0$ so as to satisfy the conditions $2_{\xi<\xi_0}$)–$5_{\xi<\xi_0}$), then in the second case, they may be determined up to b_{ξ_0} and a_{ξ_0} so that conditions $2_{\xi\leqq\xi_0}$)–$5_{\xi\leqq\xi_0}$) hold.

Proceeding in this way we finally arrive at a first ordinal number, say ξ_0, for which the first case holds, i.e. such that $A' \subseteqq \sum_{\xi<\xi_0} A'(b_\xi)$. We call B'' the set of all b_η for $\eta < \xi_0$, A'' the set of all a_η for $\eta < \xi_0$. Then conditions 1)–5) are satisfied. This completes the proof of the lemma.

We shall now formulate a definition that will be used in one of two hypotheses guaranteeing the converse of theorem II_1.

DEFINITION 9: *Two sets A' and B' such that $A' \subseteqq A$ and $B' \subseteqq B$ are said to be of the same power relative to R if there exists a one-to-one correspondence between the elements of A' and the elements of B' such that for each pair of corresponding elements $a' \in A'$ and $b' \in B'$ we have $(a', b') \in R$.*

HYPOTHESIS H_2: *If $A^* \in \mathfrak{A}^*$ and if B^* is a set $\subseteqq B$ such that A^* and B^* are of the same power relative to R, then $B^* \in \mathfrak{B}^*$.*

HYPOTHESIS H_3: *The vacuous set is an element of $\mathfrak{A}^*$.*

THEOREM II_2: *Hypotheses H_2 and H_3 imply: If A' has the property C II, then A' has the property M.*

We prove the theorem by assuming that hypotheses H_2 and H_3 hold and that A' is a set without the property M, and by showing that A' does not have the property C II.

The assumption that A' does not have the property M implies the existence of a B' satisfying the conditions

(III) $$(A', B') \in \mathfrak{R}.$$

(IV) $$B'' \subseteqq B' \text{ and } (A', B'') \in \mathfrak{R} \text{ imply } B'' \text{ non } \in \mathfrak{B}^*.$$

Let B' be well ordered:

(I) $$B' = \{b'_1, \cdots, b'_\xi, \cdots\}.$$

By virtue of the lemma, (III) implies the existence of two sets A'' and B'' such that

1) $$A'' \subseteqq A', \qquad B'' \subseteqq B', \qquad (A', B'') \in \mathfrak{R}$$

and that A'' and B'' may be well ordered

$$A'' = \{a_1, \cdots, a_\xi, \cdots\}, \qquad B'' = \{b_1, \cdots, b_\xi, \cdots\}$$

according to the following conditions:

2) $$(a_\xi, b_\xi) \in R \text{ for each } \xi,$$

3) $A''(b_\xi) \subseteqq A''_\xi$ for each ξ, where A''_ξ denotes the set of all a_η with $\eta \leqq \xi$.

From 1) and (IV) it follows that B'' *non* $\epsilon\ \mathfrak{B}^*$. On account of hypothesis H_2 and condition 2) we have thus A'' *non* $\epsilon\ \mathfrak{A}^*$. Now the set A_0'' consisting of the elements a_ξ of A'' for which $\xi < 1$, is vacuous. A_0'' and each subset of A_0'' is thus $\epsilon\ \mathfrak{A}^*$ by virtue of hypothesis H_3. Consequently, there exists a first ordinal number ξ such that at least one subset[3] of A_ξ'' is *non* $\epsilon\ \mathfrak{A}^*$. Let ξ_0 be this number. Then there exists a set A_* such that

$\alpha)$ $A_* \subseteq A''$, $\qquad \beta)$ A_* *non* $\epsilon\ \mathfrak{A}^*$,

$\gamma)$ If $\eta < \xi_0$ and $A^* \subseteq A_\eta''$, then $A^* \epsilon\ \mathfrak{A}^*$.

The set A_* has the following property: Any element a of A_* is, for some cardinal number $\eta < \xi_0$, identical with the element a_η of A_{ξ_0}''. Thus on account of conditions 2) and $\alpha)$ if follows from $a \epsilon A_*$ that $a = a_\eta \epsilon A_*(b_\eta) \subseteq A''(b_\eta)$, and hence $A_*(b_\eta) \epsilon\ \mathfrak{A}^*$ by virtue of condition $\gamma)$. Consequently, according to definition 3, the element a of A_* does not belong to A_*''. Hence $A_* \cdot A_*''$ is vacuous. Since A_* is a subset of A' which satisfies condition $\beta)$ it is shown that A' does not have the property C II. This completes the proof of theorem II_2.

If A' has the property C II, then, according to remark 4, each subset of A' has the property C II and hence under hypotheses H_2 and H_3 the property M by virtue of theorem II_2. We have thus the

COROLLARY: *Hypotheses H_2 and H_3 imply: If A' has the property C II, then each subset of A' has the property M.*

The theorems II_1 and II_2 imply the following

THEOREM II: *Under the hypotheses H_1, H_2, H_3 the mapping property M and the condensation property C II are equivalent.*

It follows from the given proof of the theorem II_2 that it may be expressed in another form which does not presuppose any assumption about a set $\mathfrak{B}^*$, and, consequently, is independent of the hypothesis H_2. We introduce the following

DEFINITION 10: *A' has the property M' if $(A', B') \epsilon\ \mathfrak{R}$ implies the existence of A^* and B^* such that $B^* \subseteq B'$, $(A', B^*) \epsilon\ \mathfrak{R}$, $A^* \epsilon\ \mathfrak{A}^*$, and that A^* and B^* have the same power relative to R.*

Then the proof of theorem II_2 yields the following

COROLLARY: *Hypothesis H_3 implies: If A' has the property C II, then A' has the property M'.*

Theorem II_2 derives the property M from the property C II, assuming hypotheses H_2 and H_3. Property C II, according to remark 3, implies property C I. In order to derive the property M from the weaker property C I we shall have to assume stronger hypotheses about the relations between the sets $\mathfrak{A}^*$ and $\mathfrak{B}^*$. An example of such an hypothesis is

Hypothesis H_4: *If $(A', B') \epsilon\ \mathfrak{R}$ and B' non $\epsilon\ \mathfrak{B}^*$, then there exists a B''' such that $B''' \subseteq B'$, $(A', B''') \epsilon\ \mathfrak{R}$, and that B''' may be well ordered $B''' = \{b_1''', \cdots,$*

[3] This point of the argument was communicated to me by Dr. A. N. Milgram in simplification of my previous proof. This modification obviated the necessity of the assumption that each subset of a set of the system $\mathfrak{A}^*$ belongs to $\mathfrak{A}^*$.

$b'''_\xi, \cdots\}$ *in such a way that each set* $\subseteq A$ *which for some* $b_\xi \in B'''$ *is of the same power relative to* R *as a subset of* B'''_ξ *is* $\in \mathfrak{A}^*$.

THEOREM I_2: *Hypothesis* H_4 *implies*: *If* A' *does not have the property* M, *then* A' *does not have the property* $C\,I$ *unless the following condition is satisfied*: *There exist three sets* A^*, B', B^* *such that* $A^* \in \mathfrak{A}^*$, $(A', B') \in \mathfrak{R}$, $(A', B^*) \in \mathfrak{R}$, $B^* \subseteq B'$, *that* A^* *and* B^* *have the same power relative to* R, *and that besides from* $B'' \subseteq B'$ *and* $(A', B'') \in \mathfrak{R}$ *it follows that* B'' *non* $\in \mathfrak{B}^*$.

In order to prove this theorem we assume that A' does not have the property M. Then there exists a set B' such that $(A', B') \in \mathfrak{R}$ and that from $B'' \subseteq B'$ and $(A', B'') \in \mathfrak{R}$ it follows that B'' *non* $\in \mathfrak{B}^*$. On account of hypothesis H_4 there exists a B''' such that $B''' \subseteq B'$, $(A', B''') \in \mathfrak{R}$, and B''' may be well ordered, $B''' = \{b'''_1, \cdots, b'''_\xi, \cdots\}$ in such a way that each set $\subseteq A$ which for some element b'''_ξ of B''' is of the same power relative to R as a subset of B'''_ξ is $\in \mathfrak{A}^*$. Applying the lemma to A' and B''' we get two sets A'' and B'' satisfying the condition

1) $$A'' \subseteq A', \qquad B'' \subseteq B, \qquad (A', B'') \in \mathfrak{R}$$

which can be well ordered, $A'' = \{a_1, \cdots, a_\xi, \cdots\}$, $B'' = \{b_1, \cdots, b_\xi, \cdots\}$ according to the following conditions

2) $(a_\xi, b_\xi) \in R$ for each ξ. (Thus A'' and B'' have the same power relative to R).

3) $A''(b_\xi) \subseteq A''_\xi + \{a_\xi\}$ for each ξ.

4) If $\eta < \xi$, $b_\eta = b'''_{\eta'''}$, $b_\xi = b'''_{\xi'''}$ then $\eta''' < \xi'''$.

Now two cases are possible:

First case: $A'' \in \mathfrak{A}^*$. Then it follows from 2) that B'' is of the same power relative to R as a set of $\mathfrak{A}^*$, viz. A''. Since according to 1) we have $(A', B'') \in \mathfrak{R}$ the condition mentioned in theorem I_2 is satisfied in this case.

Second case: A'' *non* $\in \mathfrak{A}^*$. From $(A', B'') \in \mathfrak{R}$ it follows that for each $a \in A'$ there exists a $b_\xi \in B''$ such that $a \in A''(b_\xi)$. According to 3) we have $A''(b_\xi) \subseteq A''_\xi + \{a_\xi\}$. The latter set, on account of 2) has the same power relative to R as $B'' + \{b_\xi\}$. The element b_ξ is identical with some element $b'''_{\xi'''}$ of B''' where $\xi \leqq \xi'''$ by virtue of 4). Thus $B''_\xi + \{b_\xi\} \subseteq B'''_{\xi'''}$. Since $A''(b_\xi)$ has the same power relative to R as a subset $B'''_{\xi'''}$ it follows from hypothesis H_4 that $A''(b_\xi) \in \mathfrak{A}^*$. Thus in the second case there exists a set A'' which is $\subseteq A'$ and *non* $\in \mathfrak{A}^*$ and has the following property: For each $a \in A'$ there exists a $b \in B$ such that $A''(b) \in \mathfrak{A}^*$. This completes the proof of theorem I_2.

REMARK: Hypothesis H_4 is satisfied and the condition mentioned in theorem I_2 is excluded if $\mathfrak{A}^*$ is a system of sets $\subseteq A'$ which are not of the same power as any set $B'' \subseteq B'$ satisfying the condition $(A', B'') \in \mathfrak{R}$. For such a system $\mathfrak{A}^*$ we may formulate theorem I_2 in the following way: *If* A' *has the property* $C\,I$, *then* A' *has the property* M.

Hypothesis H_4 and the condition mentioned in theorem I_2 constitute, of course, only one of many possible bases for the derivation of M from C I.

The method applied in this paper may be applied to detopologize other parts of topology as well. Theorems on well ordered monotonic sets of closed sets and other theorems usually shown as equivalent to covering theorems and to condensation properties may be treated that way and a theory of points of condensation of different powers may be developed in this direction. More complicated covering theorems[4] might be studied from this point of view. In a subsequent paper Dr. A. N. Milgram deals with other topological questions in a similar way.

It may be of interest that the results of this paper provide an answer to a question proposed to the author by W. Hurewicz in 1924: Can one prove theorems dual to the covering theorems in the following sense: Whereas the covering theorems permit one to extract from a set of sets that cover a set A (such that each point of A is contained in the interior of at least one set of the set of sets) a smaller set of sets that still covers A, one may ask whether one can extract from a set of points B that is embedded in a set of sets (such that each set of the set of sets contains in its interior at least one point of B) a smaller set of points that is still embedded in the set of sets (in the sense mentioned). If in a given space A is a set of sets, B a set of points and R the set of those pairs (a, b) for which the set a (element of A) contains the point b of B in its interior,—then obviously our theorems yield conditions under which the answer to the question is affirmative.

UNIVERSITY OF NOTRE DAME,
NOTRE DAME, INDIANA.

[4] See Ergebnisse e. mathem Kolloquiums, 2, 1930, p. 24 and the papers by Hurewicz and the author quoted there.

b_ξ''', $\cdots$ } *in such a way that each set* $\subseteq A$ *which for some* $b_\xi \in B'''$ *is of the same power relative to* R *as a subset of* B_ξ''' *is* $\in \mathfrak{A}^*$.

THEOREM I_2: *Hypothesis* H_4 *implies: If* A' *does not have the property* M, *then* A' *does not have the property* $C\,I$ *unless the following condition is satisfied: There exist three sets* A^*, B', B^* *such that* $A^* \in \mathfrak{A}^*$, $(A', B') \in \mathfrak{R}$, $(A', B^*) \in \mathfrak{R}$, $B^* \subseteq B'$, *that* A^* *and* B^* *have the same power relative to* R, *and that besides from* $B'' \subseteq B'$ *and* $(A', B'') \in \mathfrak{R}$ *it follows that* B'' *non* $\in \mathfrak{B}^*$.

In order to prove this theorem we assume that A' does not have the property M. Then there exists a set B' such that $(A', B') \in \mathfrak{R}$ and that from $B'' \subseteq B'$ and $(A', B'') \in \mathfrak{R}$ it follows that B'' *non* $\in \mathfrak{B}^*$. On account of hypothesis H_4 there exists a B''' such that $B''' \subseteq B'$, $(A', B''') \in \mathfrak{R}$, and B''' may be well ordered, $B''' = \{b_1''', \cdots, b_\xi''', \cdots\}$ in such a way that each set $\subseteq A$ which for some element b_ξ''' of B''' is of the same power relative to R as a subset of B_ξ''' is $\in \mathfrak{A}^*$. Applying the lemma to A' and B''' we get two sets A'' and B'' satisfying the condition

$$1) \qquad A'' \subseteq A', \qquad B'' \subseteq B, \qquad (A', B'') \in \mathfrak{R}$$

which can be well ordered, $A'' = \{a_1, \cdots, a_\xi, \cdots\}$, $B'' = \{b_1, \cdots, b_\xi, \cdots\}$ according to the following conditions

2) $(a_\xi, b_\xi) \in R$ for each ξ. (Thus A'' and B'' have the same power relative to R).

3) $A''(b_\xi) \subseteq A_\xi'' + \{a_\xi\}$ for each ξ.

4) If $\eta < \xi$, $b_\eta = b_{\eta'''}'''$, $b_\xi = b_{\xi'''}'''$ then $\eta''' < \xi'''$.

Now two cases are possible:

First case: $A'' \in \mathfrak{A}^*$. Then it follows from 2) that B'' is of the same power relative to R as a set of $\mathfrak{A}^*$, viz. A''. Since according to 1) we have $(A', B'') \in \mathfrak{R}$ the condition mentioned in theorem I_2 is satisfied in this case.

Second case: A'' *non* $\in \mathfrak{A}^*$. From $(A', B'') \in \mathfrak{R}$ it follows that for each $a \in A'$ there exists a $b_\xi \in B''$ such that $a \in A''(b_\xi)$. According to 3) we have $A''(b_\xi) \subseteq A_\xi'' + \{a_\xi\}$. The latter set, on account of 2) has the same power relative to R as $B'' + \{b_\xi\}$. The element b_ξ is identical with some element $b_{\xi'''}'''$ of B''' where $\xi \leqq \xi'''$ by virtue of 4). Thus $B_\xi'' + \{b_\xi\} \subseteq B_{\xi'''}'''$. Since $A''(b_\xi)$ has the same power relative to R as a subset $B_{\xi'''}'''$ it follows from hypothesis H_4 that $A''(b_\xi) \in \mathfrak{A}^*$. Thus in the second case there exists a set A'' which is $\subseteq A'$ and *non* $\in \mathfrak{A}^*$ and has the following property: For each $a \in A'$ there exists a $b \in B$ such that $A''(b) \in \mathfrak{A}^*$. This completes the proof of theorem I_2.

REMARK: Hypothesis H_4 is satisfied and the condition mentioned in theorem I_2 is excluded if $\mathfrak{A}^*$ is a system of sets $\subseteq A'$ which are not of the same power as any set $B'' \subseteq B'$ satisfying the condition $(A', B'') \in \mathfrak{R}$. For such a system $\mathfrak{A}^*$ we may formulate theorem I_2 in the following way: *If* A' *has the property* $C\,I$, *then* A' *has the property* M.

Hypothesis H_4 and the condition mentioned in theorem I_2 constitute, of course, only one of many possible bases for the derivation of M from C I.

The method applied in this paper may be applied to detopologize other parts of topology as well. Theorems on well ordered monotonic sets of closed sets and other theorems usually shown as equivalent to covering theorems and to condensation properties may be treated that way and a theory of points of condensation of different powers may be developed in this direction. More complicated covering theorems[4] might be studied from this point of view. In a subsequent paper Dr. A. N. Milgram deals with other topological questions in a similar way.

It may be of interest that the results of this paper provide an answer to a question proposed to the author by W. Hurewicz in 1924: Can one prove theorems dual to the covering theorems in the following sense: Whereas the covering theorems permit one to extract from a set of sets that cover a set A (such that each point of A is contained in the interior of at least one set of the set of sets) a smaller set of sets that still covers A, one may ask whether one can extract from a set of points B that is embedded in a set of sets (such that each set of the set of sets contains in its interior at least one point of B) a smaller set of points that is still embedded in the set of sets (in the sense mentioned). If in a given space A is a set of sets, B a set of points and R the set of those pairs (a, b) for which the set a (element of A) contains the point b of B in its interior,—then obviously our theorems yield conditions under which the answer to the question is affirmative.

UNIVERSITY OF NOTRE DAME,
NOTRE DAME, INDIANA.

[4] See Ergebnisse e. mathem Kolloquiums, 2, 1930, p. 24 and the papers by Hurewicz and the author quoted there.

ANNALS OF MATHEMATICS
Vol. 39, No. 4, October, 1938

A GENERAL EXISTENCE THEOREM AND SOME APPLICATIONS

BY A. N. MILGRAM

(Received January 27, 1938; revised June 9, 1938)

1. Introduction[1]

It is our purpose in this paper to prove some general existence theorems (Theorem 1 and Corollary 1′) concerning partially-ordered sets and relations. In section 3 some applications are given yielding such widely separated results as the Brouwer Reduction Theorem of Topology and the so-called Tonelli Principle of Calculus of Variations (a generalization of the theorem that a lower semi-continuous function defined on a compact set assumes its lower bound). In section 4 monotone collections of sets are considered and a generalization of the Cantor Product Theorem for compact sets is obtained. It must be noted that these applications do not exhaust the possible interpretations of Theorem 1 and Corollary 1′; they serve rather to indicate the methods of applying them. Probably their main interest lies in the connection they reveal between otherwise remote considerations. No use is made in their proof of transfinite numbers.

2. Existence of Extreme Elements

ASSUMPTION 1: *Let A be a set which is partially-ordered under the relation $\preceq$.*

ASSUMPTION 2: *We assume also given a denumerable sequence B of objects $b_1, b_2, \cdots, b_i, \cdots$ and a subset R of the Cartesian product[2] $A \times B$ such that if $a \preceq a'$ and $(a, b_i) \in R$ then $(a', b_i) \in R$ for any integer i.*

The following illustration will make these assumptions concrete and at the same time serve as a device by which to remember them: Let A be the set of points on the unit interval (0, 1), i.e., $a \in A$ if $0 \leqq a \leqq 1$. The ordering is to be taken such that $a \preceq a'$ is equivalent to $a \geqq a'$. Further let the rational numbers be ordered in a denumerable sequence $b_1, b_2, \cdots, b_i, \cdots$. R is taken to be the set of all pairs (a, b_i) where $b_i \geqq a$, and a is a point of A. If $a \preceq a'$ and $(a, b_i) \in R$, then from $a' \leqq a$ and $a \leqq b_i$ we have $a' \leqq b_i$ which according to this definition of R, means $(a', b_i) \in R$. Thus A and R in this case would satisfy assumptions 1 and 2.

Returning to our general considerations, we formulate several definitions.

DEFINITION 1: *If $a \in A$, we define $\rho_j(a) = \begin{cases} 1 \text{ if } (a, b_j) \in R \\ 0 \text{ otherwise} \end{cases}$ for each integer j. We then define $\sigma_i(a) = \sum_1^i \rho_j(a)$ for each integer i. In particular, $\sigma_1(a) = \rho_1(a)$.*

[1] The author is indebted to Prof. K. Menger for numerous suggestions in the preparation of this paper.

[2] By the Cartesian product $A \times B$ is meant the set of all ordered couples (a, b) where a is an element of A and b an element of B.

From Assumption 2 it follows that if $a \preceq a'$ then $\rho_j(a) \leqq \rho_j(a')$ for each integer j. This in turn implies that if $a \preceq a'$, then $\sigma_i(a) \leqq \sigma_i(a')(i = 1, 2, \cdots)$. Obviously $\sigma_i(a)$ is some non-negative integer which does not exceed i for each element a of A.

DEFINITION 2: *If A' is a subset of A, R will be said to be a separator of A' if for every pair of elements $a' \neq a''$ in A' such that $a' \prec a''$ there exists an integer k such that $\sigma_k(a') < \sigma_k(a'')$.*

(In the illustration given above R is a separator of A, and also of every subset of A. For if $a' \preceq a''$ and $a' \neq a''$, then $a'' < a'$. There exists a rational number, say b_k, between a'' and a', i.e., $a'' < b_k < a'$, so that $(a'', b_k) \in R$ and (a', b_k) non-$\in R$. Since $\rho_i(a') \leqq \rho_i(a'')$ for every i, and $\rho_k(a') = 0$, $\rho_k(a'') = 1$, we have $\sigma_k(a') < \sigma_k(a'')$).

DEFINITION 3: *A' is said to be inductive if every ordered increasing sequence $a_1 \preceq a_2 \preceq \cdots \preceq a_n \preceq \cdots$ in A' has an upper bound a^* in A'.*

DEFINITION 4: *The element a^* of A' will be said to be an extreme element of A' if for each element a' of A' the relation $a^* \preceq a'$ implies $a' = a^*$.*

THEOREM 1: *Under Assumptions 1 and 2, if A' is a subset of A, R a separator of A', and a_0 an element of A', then A' contains an ordered increasing sequence $a_0 \preceq a_1 \preceq \cdots \preceq a_n \preceq \cdots$ of elements for which each upper bound a^* is an extreme element of A'.*

PROOF: Let A_0' denote the set of elements a in A' such that $a_0 \prec a$. For any element a of A_0' we have $\sigma_1(a) = 0$ or 1. Let a_1 be an element of A_0' for which $\sigma_1(a_1) = \max_{a \in A_0'} \sigma_1(a)$. Assume we have chosen the elements $a_0, a_1, \cdots, a_{i-1}$ in A'. Let A_{i-1}' denote the set of elements a in A' such that $a_{i-1} \preceq a$ and choose a_i as an element of A_{i-1}' for which $\sigma_i(a_i) = \max_{a \in A_{i-1}'} \sigma_i(a)$. Continuing this process indefinitely we obtain the sequence

$$a_0, a_1, \cdots, a_{i-1}, a_i, \cdots, \qquad a_{i-1} \preceq a_i, \tag{2.1}$$

of elements in A' satisfying the condition that if a is an element of A' and

$$a_{i-1} \preceq a, \qquad \text{then} \qquad \sigma_i(a) \leqq \sigma_i(a_i). \tag{2.2}$$

We must now show that if a^* is an element of A' such that

$$a_i \preceq a^* \qquad (i = 0, 1, \cdots) \tag{2.3}$$

then a^* is an extreme element of A'.

Suppose to the contrary that A' contains an element $a^{**} \neq a^*$ such that $a^* \preceq a^{**}$. Then from Definition 1 and (2.3) it follows that $\sigma_j(a^{**}) \geqq \sigma_j(a^*) \geqq \sigma_j(a_i)$ for all integers i and j. From the assumption that R is a separator of A', we may conclude that there exists an integer k such that $\sigma_k(a^{**}) > \sigma_k(a^*)$ (see Def. 2). From (2.3) and $a^* \preceq a^{**}$ we have

$$a_{k-1} \preceq a^* \qquad \text{and} \qquad \sigma_k(a^{**}) > \sigma_k(a_k)$$

which contradicts (2.2). Hence the assumption that any element a^* which satisfies (2.3) is not an extreme element leads to a contradiction. The sequence (2.1) satisfies the statement of the theorem.

COROLLARY 1: *If A' is a subset of A inductive relative to T, R a separator of A', and a_0 an element of A', then A' contains an extreme element a^* such that $a_n \preceq a^*$ $(n = 0, 1, 2, \cdots)$ where $a_0, a_1, \cdots$ is the sequence referred to in Theorem* 1.

If A' is inductive, there must exist an element a^* such that $a_n \preceq a^*$ for every integer n, and from Theorem 1 it follows that a^* is an extreme element of A'.

The statement of Corollary 1 in the form in which it will frequently be used does not involve the elements $a_1, a_2, \cdots, a_n, \cdots$, i.e.,

COROLLARY 1′: *Under Assumptions* 1 *and* 2, *if A' is an inductive subset of A, R a separator of A', and a_0 an element of A', then A' contains an extreme element a^* such that $a_0 \preceq a^*$.*

COROLLARY 2: *If A' is such that $\rho_i(a) = \rho_i(a')$ for every integer i implies $a = a'$, then under the assumptions of Corollary* 1, *a^* is unique.*

PROOF: Let a^{**} be an element of A' for which $a_n \preceq a^{**}$ for every n. We shall show that $\rho_i(a^{**}) = \rho_i(a^*)$ for every integer i. Suppose to the contrary there exists an integer j such that $\rho_j(a^{**}) \neq \rho_j(a^*)$. Let k be the first integer for which $\rho_k(a^{**}) \neq \rho_k(a^*)$. If $\rho_k(a^{**}) > \rho_k(a^*)$, then $\sigma_k(a^{**}) > \sigma_k(a^*)$. But $\sigma_k(a^*) \geqq \sigma_k(a_k)$ since $a_k \preceq a^*$. Therefore $\sigma_k(a^{**}) > \sigma_k(a_k)$, and this together with $a_k \preceq a^{**}$ violates condition (2.2). Hence we cannot have $\rho_k(a^{**}) > \rho_k(a^*)$, and by a similar argument we cannot have $\rho_k(a^*) > \rho_k(a^{**})$. From this it follows that $\rho_j(a^*) = \rho_j(a^{**})$ for every integer j, and therefore, under the hypothesis of the corollary, that $a^* = a^{**}$. Q.E.D.

3. Applications

1. Let S be a topological space with a countable basis $U_1, U_2, \cdots$. The following theorems are re-statements of Corollary 1′ after A, B, $\overline{\prec}$, R and A' have been specified.

THEOREM 2: (*Brouwer Reduction Theorem*) *Let E_1 be a property of closed sets in S such that if $F_1 \supseteq F_2 \supseteq \cdots \supseteq F_n \supseteq \cdots$ are sets having property E_1, then $\prod_{n=1}^{\infty} F_n$ has property E_1. If F is any set in S having property E_1, then F contains an irreducible subset having property E_1.*

Let A be the class of all subsets of S. $a \preceq a'$ if and only if $a \supseteq a'$. $(a, b_i) \in R$ if and only if $a \subseteq S - U_i$. The set A' of all closed sets in S having property E_1 is an inductive subset and R is a separator of A'. That A' is inductive is contained in the hypothesis of Theorem 2, and that R is a separator of A' follows from the fact that if F_1 and F_2 are any two closed sets in S and F_2 is a proper subset of F_1, there exists an integer i such that $F_1 \nsubseteq S - U_i$ and $F_2 \subseteq S - U_i$. Thus reading a_0 as F, Corollary 1′ becomes Theorem 2.

THEOREM 3: *Let E_2 be a property of closed sets in S such that if $F_1 \subseteq F_2 \subseteq \cdots \subseteq F_i \subseteq \cdots$ is a sequence of sets having property E_2, then $\overline{(\sum_i^{\infty} F_i)}$ has property E_2. If F is any set in S having property E_2, then F is contained in a maximal set having property E_2.*

Let A and B be as in Theorem 2. $a \preceq a'$ if and only if $a \subseteq a'$. $(a, b_i) \in R$ if and only if $a \cdot U_i \neq 0$. The set A' of all closed sets in S having property E_2

is an inductive subset, and R is a separator of A'. This follows from the hypothesis on E_2 and the fact that if F_1 and F_2 are two closed subsets of S and F_1 is a proper subset of F_2, there exists an integer i such that $F_1 \cdot U_i = 0$ and $F_2 \cdot U_i \neq 0$. Thus again reading a_0 as F, Corollary 1′ becomes Theorem 3.

2. Similar theorems for open sets may be stated, one dealing with a property preserved under summation, and the other with a property preserved by the open kernel of the product.

3. Another interpretation of Corollary 1′ which is of interest not because it provides a simple proof, but because it reveals another bond between certain concepts in Topology and Analysis is

THEOREM 4: *(Tonelli's Principle) Let $C_1, C_2, \cdots, C_n, \cdots$ be a sequence of compact sets contained in the Fréchet space L^* and $\beta(x)$ a function defined on $\sum_1^\infty C_n$, which is lower semi-continuous on each of the sets C_n. We assume further that for each integer n there exists an integer m_n such that $C_{m_n} \supseteq L^*[\beta(x) \leqq n]$. Then every closed subset F of $\sum_1^\infty C_n$ contains an element x_F where $\beta(x)$ assumes its greatest lower bound for x in F.*[3]

PROOF: We may suppose there exists an integer N such that $M = F[\beta(x) \leqq N]$ is non-vacuous (otherwise $\beta(x)$ would be infinite at each point x and any point would serve as x_F). For each number $s \leqq N$ we denote by M_s the set of points in M which are mapped on s by $\beta(x)$. We define A as the class of sets M_s $(-\infty \leqq s \leqq N)$, though for convenience we shall continue to call the elements of A, as before, $a, a', a'', \cdots$ $a \preceq a'$ if and only if $\beta(a) \geqq \beta(a')$ (where $\beta(a)$ is the number on which every point of a is mapped). Let $r_1, r_2, \cdots$ denote the rational numbers in a definite order, then we define $(a, b_i) \epsilon R$ if and only if $\beta(a) \leqq r_i$. R is a separator of A since $a \neq a'$ and $a \preceq a'$ implies that $\beta(a) > \beta(a')$. Hence we can find an integer i such that $\beta(a) > r_i > \beta(a')$, i.e., (a, b_i) non-ϵR and $(a', b_i) \epsilon R$. Moreover A is inductive. To prove this we consider a sequence of elements $a_0, a_1, \cdots, a_i, \cdots$ in A such that $a_i \preceq a_i + 1$. Since

[3] K. Menger in his paper "*Die metrische Methode in der Variationsrechnung,*" Ergebnisse eines mathematischen Kolloquiums, v.8, 1937, p. 13, called Tonelli's Principle the following theorem:

On the Fréchet space L^* let $\alpha(x)$ be a (not necessarily finite valued) function. Let a finite valued function $\beta(x)$ be defined on the set $L^*[\alpha(x) < \infty]$ such that for each finite number ρ the following conditions are satisfied:

1) The set $L^*[\alpha(x) \leqq \rho]$ is compact.

2) $\beta(x)$ is lower semi-continuous on $L^*[\alpha(x) \leqq \rho]$

3) $\alpha(x)$ is bounded from above on the set $L^*[\beta(x) \leqq \rho]$

Under these conditions every closed subset A of $L^*[\alpha(x) < \infty]$ contains at least one element x_A such that $\beta(x_A)$ is the greatest lower bound for $\beta(x)$ on A.

(Tonelli had used this theorem in the special case where L^* is a space whose elements are rectifiable arcs and $\alpha(x)$ is the arc-length). The above statement (i.e. Theorem 4) includes that given by Menger as a special case. By letting ρ in 1) take the values $1, 2, \cdots, n, \cdots$ we obtain the sequence of compact sets $C_n = L^*[\alpha(x) \leqq n]$ which are however monotonically non-decreasing, a condition not necessary to the statement of the theorem. To require that $C_n = L^*[\alpha(x) \leqq n]$ is compact for $n = 1, 2, \cdots$ is not so strong a restriction as that in 1).

$C_{m_N} \supseteq M$, the sets $a_0, a_1, \cdots$ are subsets of the compact set C_{m_N}, and since F is closed, $F \cdot C_{m_N}$ must contain a point p which belongs to the limit superior[4] of $a_0, a_1, \cdots$. From the lower semi-continuity of $\beta(x)$ on C_{m_N} and the monotonicity of the sequence $\beta(a_0) \geqq \beta(a_1) \geqq \beta(a_2) \cdots$ it follows that $\beta(p) \leqq \beta(a_i)$, $i = 0, 1, \cdots$. The set a' which contains p is such that $a_i \preceq a'$ for every i. This completes the demonstration that A is inductive. If a_0 is any element of A, then from Corollary 1', A contains an extreme element a^* and any point of a^* will serve as the x_F in the statement of the theorem.

4. Monotone Collections of Sets

A collection of sets $\{C\}$ is monotone if for every pair of sets C and C' in the collection either $C \supseteq C'$ or $C' \supseteq C$. In this section we shall prove several theorems for monotone collections. It must be noted that while every monotonically non-increasing (non-decreasing) countable or transfinite sequence is a monotone collection the converse does not necessarily hold (as the simple illustration $(-x, x)$, $0 < x < 1$, shows).

DEFINITION: *If C is a totally bounded[5] space we denote by n_C the minimum number of open sets of diameter less than $1/n$ with which it is possible to cover C.*

LEMMA: *If $C' \supset C$, C a proper subset of C', are two totally bounded sets, then* 1) *$n_{C'} \geqq n_C$ for every n, and* 2) *there exists an integer N such that for $m > N$, $m_{C'} > m_C$.*

PROOF: The proof of 1) is contained in the observation that any covering of C' is also a covering of C. Now let p be a point of $C' - C$. Any integer N such that $1/N < \frac{1}{2}\rho(p, C)$ satisfies 2) of the lemma. For let $U_1, U_2, \cdots, U_{m_{C'}}$ be a minimum covering of C' with sets of diameter less than $1/m$, where $m \geqq N$. If we extract from these the sets which meet C, say $U_{i_1}, U_{i_2}, \cdots, U_{i_k}$ we obtain a covering of C. $k \leqq m_{C'} - 1$, since at least one set occurring in $U_1, \cdots, U_{m_{C'}}$, namely a set which contains p, does not meet C and hence is not one of the sets $U_{i_1}, \cdots, U_{i_k}$. But $k \geqq m_C$, since $U_{i_1}, \cdots, U_{i_k}$ is a covering of C, so that $m_{C'} - 1 \geqq m_C$.

THEOREM 5: *If in a metric space S, $\{a\}$ denotes a monotone collection of totally bounded sets, then $\sum a$, where the summation is taken over all $\{a\}$, is the sum of countably many of the sets of $\{a\}$.*

PROOF: With each set a' of $\{a\}$ we associate the number $K_{a'} = (1_{a'} + 1)^{-2} + (2_{a'} + 2)^{-2} + \cdots + (n_{a'} + n)^{-2} + \cdots$. Let A be the set of elements of $\{a\}$ and $a \preceq a'$ if $a \subseteq a'$. If $a' \supseteq a$, then as a direct consequence of our lemma we have $K_{a'} < K_a$. Ordering the rational numbers in a sequence $r_1, r_2, \cdots,$

[4] A point p belongs to the limit superior of the sequence of sets $a_0, a_1, \cdots$ if every neighborhood of p contains points of infinitely many of these sets. See Kuratowski, Topologie, p. 153.

[5] A metric space is said to be totally bounded if for every $\epsilon > 0$ it may be expressed as the sum of a finite number of open sets each of diameter less than ϵ. A compact metric space is always totally bounded.

we define $(a, b_1) \in R$ if $K_a \leqq r_i$. R is a separator of A, for from $a \neq a'$, $a \preceq a'$ we have $K_{a'} < K_a$, and hence choosing an i such that $K_{a'} < r_i < K_a$ we shall have $(a', b_i) \in R$ and (a, b_i) non-$\in R$. Therefore, by Theorem 1 there exists a sequence $a_0, a_1, \cdots, a_n, \cdots$ such that $a_n \preceq a_{n+1}$, i.e. that $a_{n+1} \supseteq a_n$. Furthermore if there exists an element a^* of A such that $a_n \preceq a^*$ for every n, then a^* is an extreme element of A. This would mean that there is no set a of A such that $a \supset a^*$, and hence for every set a of A we must have $a^* \supseteq a$. Therefore the existence of a^* would imply $a^* \supseteq \sum a$ and since a^* is itself an element of A, this may be replaced by the equality $a^* = \sum a$. Thus in this case $\sum a$ would certainly satisfy the statement of the theorem.

If there exists no element a^* such that $a_n \preceq a^*$ for every n, then $\sum a = \sum_0^\infty a_n$. For if a is any element in A, we do not have $a_n \preceq a$ for every n by assumption. If therefore n' is an integer such that $a_{n'}$ non $\preceq a$, i.e. $a \not\ni a_{n'}$, then since A is a monotone collection $a_{n'} \supset a$. Consequently $\sum_0^\infty a_n \supset a$ for each a in A, and from this we obtain $\sum_0^\infty a_n \supseteq \sum a$. Since the a_n are elements of A this inclusion must be the equality $\sum_0^\infty a_n = \sum a$. Q.E.D.

THEOREM 6: (*Generalized Cantor Intersection Theorem*)[6] *In a metric space if* $\{a\}$ *is a monotone collection of compact sets, then* Πa, *where the product is taken over all* a *in* $\{a\}$, *is compact.*

PROOF: Let A denote the class of elements in $\{a\}$ and $a \preceq a'$ if and only if $a \supseteq a'$. Let $(a, b_i) \in R$ if and only if $K_a \geqq r_i$. As before, in Theorem 5, it may be shown that R is a separator of A, so that again applying Theorem 1 we obtain the sequence $a_0, a_1, \cdots, a_n, \cdots a_n \preceq a_{n+1}$ of elements in A.

If there exists an element a^* of A such that $a_n \prec a^*$ for every integer n, a^* would be an extreme element of A relative to T. This would mean that no set of $\{a\}$ would be contained in a^* as a proper subset, and hence from the monotonicity of $\{a\}$, $a \supseteq a^*$ for each set a of $\{a\}$. Thus we should have $\prod a = a^*$ is compact.

On the other hand, if there exists no extreme element of A, it is evident that if a is an element of A there exists an integer n such that $a \supseteq a_n$. Hence $a \supseteq \prod_{n=0}^{\infty} a_n$, from which $\prod a \supseteq \prod_{n=0}^{\infty} a_n$. Since a_n is an element of $\{a\}$, $a_n \supseteq \prod a$, so that $\prod_{n=0}^{\infty} a_n \supseteq \prod a$. Therefore $\prod_{n=0}^{\infty} a_n = \prod a$. Since the product of a countable monotonically non-increasing sequence of non-vacuous compact sets is a non-vacuous compact set, it follows that $\prod_{n=0}^{\infty} a_n$ and hence $\prod a$ is a non-vacuous compact set.

REMARKS: As noted above, the interpretations of Theorem 1 and Corollary 1′ are by no means restricted to those given here. For example, a restatement of

[6] This may also be deduced from a theorem of R. L. Moore, Proceeding of the National Academy, vol. 10, 1924.

Corollary 1′ as a theorem in Metric Geometry may be given.[7] It should be mentioned that once having chosen R, the b_i's may always be considered as subsets of A with the condition that $a \in b_i$ if and only if $\rho_i(a) \neq 0$. Thus we should have $(a, b_i) \in R$ if and only if $a \in b_i$. The important step in any application of Theorem 1 and Corollary 1′ lies in the choice of R so as to make it a separator of a previously given subset A' of A (which might be A itself). Thus if A' is an inductive subset of A, it is sufficient for the existence of an extreme element of A' relative to T that R may be so chosen. However it is not necessary, as may be verified by an example.

UNIVERSITY OF NOTRE DAME,
NOTRE DAME, INDIANA.

[7] This will appear in a note written jointly by Prof. K. Menger and the present author to be published in the Bull. of the Am. Math. Soc.

ANNALS OF MATHEMATICS
Vol. 39, No. 4, October, 1938

METRIC SPACES AND COMPLETELY MONOTONE FUNCTIONS[1]

BY I. J. SCHOENBERG

EDMUND LANDAU IN MEMORIAM

(Received May 31, 1938)

INTRODUCTION

A set $\mathfrak{S}$ of elements, or points, $P, P', \cdots$, is said to be a semi-metric space if it is provided with a distance function PP' with the following two properties: 1. $PP' = P'P \geqq 0$, 2. $PP' = 0$ if and only if $P = P'$. Let now $F(t)$ be a real continuous function defined for $t \geqq 0$, $F(t) > 0$ if $t > 0$, $F(0) = 0$. According to L. M. Blumenthal we call the new semi-metric space which arises if we replace the metric PP', of $\mathfrak{S}$, by $F(PP')$ the *metric transform* of $\mathfrak{S}$ by $F(t)$ and denote it by the suggestive symbol $F(\mathfrak{S})$ ([2], p. 8).[2] This concept was also investigated by W. A. Wilson ([21], p. 64). Certain metric transforms have long been in use, for it is well known that if $\mathfrak{S}$ is metric, also the space with the new distance function $d(P, P') = PP'/(1 + PP')$ is metric. Moreover, these two metrics are topologically equivalent and the new space has a finite diameter. The new space is clearly the metric transform $F(\mathfrak{S})$ by the function

$$F(t) = \frac{t}{1+t}. \tag{0.1}$$

A general type of question suggested by the concept of metric transforms is whether functions $F(t)$ may be found such that $\mathfrak{S}$, if metrically transformed by $F(t)$, becomes more nearly comparable to certain simple spaces; or whether non-trivial functions $F(t)$ exist which will preserve certain simple characteristics of the original space $\mathfrak{S}$. To illustrate these points we shall state a few recent results.

If $\mathfrak{S}$ is a metric space it is readily seen that the metric transform $(\mathfrak{S})^\kappa$ (of $\mathfrak{S}$ by the function $F(t) = t^\kappa$) is also metric, provided the exponent κ is chosen in the range $0 < \kappa \leqq 1$. Blumenthal has shown that if this exponent is further restricted within the range $0 < \kappa \leqq \frac{1}{2}$, then $(\mathfrak{S})^\kappa$ has the 4-point property, i.e., any *four* points of $(\mathfrak{S})^\kappa$ may be imbedded[3] in a euclidean space. Hence $(\mathfrak{S})^\kappa$ $(0 < \kappa \leqq \frac{1}{2})$ is more "nearly euclidean" as compared with $\mathfrak{S}$, for $\mathfrak{S}$ enjoys only the 3-point property which is equivalent to the triangle inequality (see [2]).

As an example of a metric transformation which preserves certain properties

[1] Presented to the American Mathematical Society, April 16, 1938.

[2] The numbers in square brackets refer to the list of references at the end of this paper.

[3] Here and throughout this paper the word *imbedding* is meant in the sense of *isometric imbedding*.

of a space we mention the result of Wilson ([21], p. 64) to the effect that the metric transform $(E_1)^{\frac{1}{2}}$, of E_1[4] by the function $t^{\frac{1}{2}}$, which is not any more a euclidean space, is still isometrically imbeddable in $\mathfrak{H}$. This result was recently extended in two different directions. 1. It was shown by the author ([15]) that $(\mathfrak{H})^{\kappa}$ $(0 < \kappa \leqq 1)$ is imbeddable in $\mathfrak{H}$.[5] More significant is the following second result. 2. John von Neumann and the author ([12]) have determined *all* those functions $F(t)$ with the property that the metric transform $F(E_1)$ be imbeddable in $\mathfrak{H}$, or that it be imbeddable in E_n. This problem is equivalent with the problem of determining all screw lines in Hilbert space and in euclidean spaces.

These developments suggest the following general problem: *To determine all functions $F(t)$ such that the metric transform $F(E_m)$ be isometrically imbeddable in E_n, where the dimensions m and n are given in advance* $(1 \leqq m \leqq \infty, 1 \leqq n \leqq \infty)$.

We shall refer to this as the problem $\{E_m ; E_n\}$. The problems of the screw lines in E_n and $\mathfrak{H}$ are identical with the problems $\{E_1 ; E_n\}$ and $\{E_1 ; \mathfrak{H}\}$ in the present notation. The existence of *closed* screw lines in $E_n (n \geqq 2)$ and in $\mathfrak{H}$ may suggest that interesting solutions of a problem $\{E_m ; E_n\}$ are omitted by restricting the transforming function $F(t)$ to be > 0 if $t > 0$. However, if $F(t)$ does vanish for positive values of t, then $F(E_m)$ is by no means a semi-metric space. However, the problem of isometric imbedding of $F(E_m)$ in E_n retains also in this case its old meaning in an obvious way. In the present paper we are primarily interested in the case when $m \geqq 2$ and a simple lemma (Lemma 6, §5) will show that if we exclude the trivial solution $F(t) \equiv 0$, a problem $\{E_m ; E_n\}$, where m exceeds one, does admit, if any, *only* solutions $F(t)$ which are > 0 for $t > 0$.

The case when $m > n$ is immediately ruled out as trivial, for a moment's reflection shows that $F(t) \equiv 0$ is the only solution of $\{E_m ; E_n\}$. In the Appendix, §7, we solve the problem $\{E_m ; E_n\}$, where $2 \leqq m \leqq n < \infty$, with the rather negative result that the obvious solutions $F(t) = ct$ $(c \geqq 0)$, which correspond to similitude transformations of E_m, are the only possible ones. The problems $\{E_m ; \mathfrak{H}\}$, where $2 \leqq m < \infty$, will be partially solved in §8 of the Appendix where all "rectifiable" solutions of these problems are determined by a method suggested by recent investigations on homogeneous random processes in the calculus of probabilities. The chief objective of this paper is a complete solution of the problem $\{\mathfrak{H}; \mathfrak{H}\}$ (Part II, §5). The solutions $F(t)$ of this problem, as described by Theorems 6 and 6′ of §5, turn out to be essentially integrals of completely monotone functions. In particular it is found that (0.1) is a solution of this problem. Thus $F(\mathfrak{H})$, where $F(t) = t/(1 + t)$, is not only a bounded homeomorph of $\mathfrak{H}$, but is actually congruent to a subset of $\mathfrak{H}$.

A convenient way of attacking analytically these geometrical problems is afforded by combining an imbedding theorem of Menger ([9]) with a connection

[4] We denote by E_m the m-dim. euclidean space, while E_∞ or $\mathfrak{H}$ shall mean the real Hilbert space.

[5] This is a special case (for $p = 2$) of the following result: $(L_p)^{\kappa} (0 < \kappa < p/2; 0 < p \leqq 2)$ is imbeddable in $\mathfrak{H}$.

between euclidean simplices and quadratic forms, which goes back to Gauss, and which is so familiar in the geometry of numbers. The author had previously emphasized the usefulness of this combination in problems of isometric imbedding in Hilbert space ([14], [15], [16]). It allows us to attack our problems with such powerful tools as the Fourier-Stieltjes and Laplace-Stieltjes integrals.

Part I of this paper, on which the subsequent work is based, incidentally answers a question which has puzzled the author long before he ever thought of metric transforms of spaces, namely the connection between the class $\mathfrak{P}$ of Fourier-Stieltjes integrals (positive definite functions)

$$g(t) = \int_{-\infty}^{\infty} e^{itu}\, d\alpha(u), \qquad (-\infty < t < \infty), \tag{0.2}$$

and the class $\mathfrak{M}$ of Laplace-Stieltjes integrals (completely monotone functions)

$$f(t) = \int_{0}^{\infty} e^{-tu}\, d\beta(n), \qquad (0 \leqq t < \infty), \tag{0.3}$$

where $\alpha(u)$ and $\beta(u)$ are bounded non-decreasing functions. In spite of the entirely different analytical character of these two classes,[6] a certain kinship was to be expected for the following two reasons: 1. In both cases the defining kernel is the exponential function. 2. The less formal reason of the similarity of the closure properties of both classes, for both classes are *convex*, i.e., $a_1 f_1 + a_2 f_2$ ($a_1 \geqq 0$, $a_2 \geqq 0$) belongs to the class if f_1 and f_2 belong to it, *multiplicative*, i.e., also $f_1 \cdot f_2$ belongs to the class, and finally *closed* with respect to ordinary convergence to a continuous limit function. The answer, as given by Theorem 3 below, is based on a result of S. Bochner on characteristic functions of distribution functions in E_m([4], p. 407) and may best be stated in terms of such functions. A subclass of the class of characteristic functions $f(x_1, \cdots, x_m)$ in E_m (defined by (1.4) below) are those which are of radial symmetry. These are completely described by Theorem 1 below. While there is not as yet at present an explicit determination of the characteristic functions in Hilbert space $\mathfrak{H}$ by an appropriate Stieltjes integration process over $\mathfrak{H}$, these functions may easily enough be defined as positive definite vector functions in $\mathfrak{H}$. Those positive definite vector functions in $\mathfrak{H}$ which are of *radial symmetry* are now explicitly determined and turn out to be identical with the functions $f(t^2)$, where $f(t) \in \mathfrak{M}$. Incidentally, Theorem 3 exhibits a new property (3.3) of completely monotone functions which is sufficient to characterize this class of functions.

The sixth section of Part II is devoted to further applications to the theory of completely monotone functions. It is shown that the solution of the problem $\{\mathfrak{H}; \mathfrak{H}\}$ (Theorem 6) is equivalent with a complete description of those trans-

[6] Indeed $f(t)$, defined by (0.3), is analytic in the half-plane $\Re t > 0$, while Weierstrass' nowhere differentiable function $g(t) = \sum_0^\infty a^n \cos b^n t$, $(0 < a < 1, ab \geqq 1)$, is readily representable as a Fourier integral (0.2) with an appropriate step function $\alpha(t)$. Thus $\mathfrak{P}$ contains "arbitrary" functions.

formations of the variable $t \mid \phi(t)$ which turn completely monotone functions $f(t)$ into functions $f(\phi(t))$ of the same class (Theorem 8). We venture to call them *inner transformations* of the class $\mathfrak{M}$. The fact that an essential part of this class of transformations does have this property was pointed out quite recently by S. Bochner ([5], p. 498).

I. Positive Definite Functions and Completely Monotone Functions

1. Positive definite functions in euclidean space E_m

1.1. Let $\mathfrak{S}$ be an abstract set of elements $P, P', Q, \cdots$. A real or complex-valued function $F(P, Q)$ of two arbitrary points of $\mathfrak{S}$ is called positive definite, if it enjoys the following two properties:

1. Hermitean symmetry

$$F(P, Q) = \overline{F(Q, P)}. \tag{1.1}$$

2. For any n points $P_1, \cdots, P_n$ of $\mathfrak{S}(n = 2, 3, \cdots)$ we have

$$\sum_{j,k=1}^{n} F(P_j, P_k)\rho_j \bar{\rho}_k \geqq 0 \tag{1.2}$$

for arbitrary ρ_j.[7]

If $\mathfrak{S}$ is also endowed with a metric PP', we would also require our function $F(P, Q)$ to be continuous in both points with respect to this metric. By imposing successive restrictions on $\mathfrak{S}$, as well as $F(P, Q)$, we shall now examine certain subclasses of Moore's general class of positive definite functions.

Let us assume first that $\mathfrak{S}$ is a linear vector space with the norm (metric) $\mid P - P' \mid = PP'$. In this case we limit ourselves to positive definite functions $F(P, Q)$ which are functions of the vector $P - Q$ only:

$$F(P, Q) = f(P - Q). \tag{1.3}$$

Our continuous vector function $f(P - Q)$ is now subject to the conditions

$$f(P - Q) = \overline{f(Q - P)}, \tag{1.1'}$$

$$\sum_{j,k=1}^{n} f(P_j - P_k)\rho_j \bar{\rho}_k \geqq 0. \tag{1.2'}$$

If our underlying space is the m-dimensional euclidean space E_m, these functions $f(P - Q)$ are precisely the positive definite functions of m variables as defined by M. Mathias ($m = 1$) and S. Bochner (m arbitrary) ([4], p. 406). Let us refer E_m to rectangular coördinates and write $P = x = (x_1, \cdots, x_m)$, $f(P - 0) = f(x_1, \cdots, x_m) = f(x)$, $xy = x_1y_1 + \cdots + x_my_m$. Bochner established the identity of this class of positive definite functions with the class of characteristic functions of distribution functions in E_m:

$$f(x) = \int_{E_m} e^{ixy}\, d\phi(y), \tag{1.4}$$

[7] This general concept is due to E. H. Moore. See [11], pp. 3–4, 173, 181–190, 209–220. Moore calls $F(P, Q)$ a positive Hermitean matrix.

where $\phi(y)$ is the monotone point function of a non-negative, bounded, totally additive set function $\phi(S)$ defined for all Borel sets of E_m.

A subclass of the class of vector functions $f(P - Q)$ associated with $\mathfrak{S}$ and enjoying the properties (1.1′) and (1.2′), is made up of those functions which are functions of the length $| P - Q |$ of the vector only:

$$f(P - Q) = g(| P - Q |) = g(PQ), \tag{1.5}$$

where $g(t)$ is a continuous function defined for $t \geqq 0$. They are necessarily real, for (1.5) implies the ordinary symmetry $f(P - Q) = f(Q - P)$, which together with the hermitean symmetry (1.1′) obviously implies the reality of $f(P - Q)$ and $g(t)$. These real functions $g(t)$ will be called positive definite "in $\mathfrak{S}$" throughout this paper. This final restriction implies on the other hand a gain in generality, for this restricted definition of positive definite functions may at once be extended to any semi-metric space $\mathfrak{S}$.

DEFINITION 1. *Let $\mathfrak{S}$ be a semi-metric space with the metric PP'. A real continuous function $g(t)$ defined in the range of values of $t = PP'(P, P' \in \mathfrak{S})$, is said to be positive definite in $\mathfrak{S}$, if for any n points $P_1, \cdots, P_n$ of $\mathfrak{S}(n = 2, 3, \cdots)$ we have*

$$\sum_{j,k=1}^{n} g(P_j P_k)\rho_j \rho_k \geqq 0, \tag{1.2″}$$

for arbitrary real ρ_j. We shall denote this class of functions by the symbol $\mathfrak{P}(\mathfrak{S})$.[8]

The class $\mathfrak{P}(\mathfrak{S})$ is never empty, for $g(t) \in \mathfrak{P}(\mathfrak{S})$, if $g(t) \equiv 1$. From (1.2″) we conclude that $| g(t) | \leqq g(0)$. It was pointed out before ([16], §2) that the class $\mathfrak{P}(\mathfrak{S})$ is convex, multiplicative and closed with respect to convergence to a continuous limit function.

1.2. Let us determine the functions $g(t)$ of the positive definite class $\mathfrak{P}(E_m)$. From (1.5) and Bochner's theorem we learn that our $g(t)$ are in a one-to-one correspondence with those characteristic functions (1.4) which are functions of $| x |$ only, i.e., of radial symmetry, and that the relationship valid both ways is

$$f(x) = g(| x |). \tag{1.5′}$$

Let $d\omega(\xi)$ denote the area element of the spherical shell $| \xi | = 1$ in E_m, and $\omega_m = \int d\omega(\xi)$ be its total area. The mean value of $e^{ix\xi}$ over $| \xi | = 1$ is obviously invariant with respect to rotations in E_m about the origin and hence is a function of $| x |$ only. We write for it

$$M_\xi\{e^{ix\xi}\} = \frac{1}{\omega_m} \int_{|\xi|=1} e^{ix\xi}\, d\omega(\xi) = \Omega_m(| x |). \tag{1.6}$$

By m-dimensional polar coördinates we readily find

$$\Omega_m(r) = \int_0^\pi e^{ir\cos\theta} \sin^{m-2}\theta\, d\theta \Big/ \int_0^\pi \sin^{m-2}\theta\, d\theta, \qquad (m \geqq 2), \tag{1.7}$$

[8] We see that $g(t) \in \mathfrak{P}(\mathfrak{S})$ if and only if $F(P, Q) = g(PQ)$ is positive definite in the sense of Moore.

whence the power series expansion and expression in terms of Bessel functions

$$\begin{aligned}\Omega_m(r) &= 1 - \frac{r^2}{2m} + \frac{r^4}{2\cdot 4\cdot m(m+2)} - \frac{r^6}{2\cdot 4\cdot 6m(m+2)(m+4)} + \cdots \\ &= \Gamma\left(\frac{m}{2}\right)\left(\frac{2}{r}\right)^{\frac{1}{2}(m-2)} J_{\frac{1}{2}(m-2)}(r), \qquad (m \geqq 1),\end{aligned} \tag{1.8}$$

showing that $\Omega_m(r)$ is an integral transcendental function.[9] We may now readily derive the following theorem.

THEOREM 1. *The class $\mathfrak{P}(E_m)$ of functions which are positive definite in E_m is identical with the class of functions of the form*

$$g(t) = \int_0^\infty \Omega_m(tu)\, d\alpha(u), \tag{1.9}$$

where $\alpha(u)$ is non-decreasing and bounded for $u \geqq 0$, and $\Omega_m(r)$ is the integral function defined above.

Indeed, let $f(x)$ of (1.4) be of radial symmetry: $f(x) = g(|x|)$. Since $f(x) = f(|x|\xi)$, for all unit vectors ξ, we have

$$\begin{aligned} g(|x|) = f(x) &= \frac{1}{\omega_m}\int_{|\xi|=1} f(|x|\xi)\, d\omega(\xi) \\ &= \frac{1}{\omega_m}\int_{|\xi|=1}\left(\int_{E_m} e^{i|x|\cdot\xi y}\, d\phi(y)\right) d\omega(\xi) \\ &= \int_{E_m}\left(\frac{1}{\omega_m}\int_{|\xi|=1} e^{i|x|\cdot y\xi}\, d\omega(\xi)\right) d\phi(y) \\ &= \int_{E_m} \Omega_m(|x|\cdot|y|)\, d\phi(y) = \int_0^\infty \Omega_m(|x|u)\, d\alpha(u),\end{aligned} \tag{1.10}$$

[10]

where we put

$$\alpha(u) = \int_{|y|\leqq u} d\phi(y) = \phi(|y| \leqq u).$$

Thus (1.10) shows that functions of $\mathfrak{P}(E_m)$ are of the form (1.9). The converse is obvious, for $\Omega_m(|x|u)$ is a characteristic function on account of (1.6).

2. Positive definite functions in Hilbert space $\mathfrak{H}$

2.1. Definition 1 implies a certain monotoneity property of the class $\mathfrak{P}(\mathfrak{S})$ in its dependence on the space $\mathfrak{S}$, for if $\mathfrak{S}$ is a subset of a space $\mathfrak{S}'$, or congruent to a subset of $\mathfrak{S}'$, then $\mathfrak{P}(\mathfrak{S}') \subset \mathfrak{P}(\mathfrak{S})$. Now as $E_1 \subset E_2 \subset \cdots \subset E_m \subset \cdots \subset \mathfrak{H}$, we see that

$$\mathfrak{P}(E_1) \supset \mathfrak{P}(E_2) \supset \cdots \supset \mathfrak{P}(E_m) \supset \cdots \supset \mathfrak{P}(\mathfrak{H}),$$

[9] The $\Omega_m(r)$ may be called Poisson functions as they were discovered by Poisson in connection with problems on heat conduction prior to Bessel's investigation of his functions. See G. N. Watson [18], p. 24.

[10] The interchange of integrations in (1.10) is justified by a general argument found in [4], p. 393.

with $\mathfrak{P}(\mathfrak{H})$ actually identical with the intersection (logical product) of this non-increasing sequence of classes $\{\mathfrak{P}(E_m)\}$.

It was pointed out elsewhere ([16], §2) that $g(t) = \exp\{-t^2u^2\}$ (u a real constant) is positive definite in E_m, for all m, hence also positive definite in $\mathfrak{H}$. This fact is immediately apparent from the elementary formula

$$e^{-(x_1^2+\cdots+x_m^2)u^2} = (2u)^{-m}\pi^{-m/2}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{i(x_1y_1+\cdots+x_my_m)}e^{-(y_1^2+\cdots+y_m^2)/4u^2}\,dy_1\cdots dy_m, \tag{2.1}$$

showing that $f(x) = \exp\{-|x|^2u^2\}$ is a characteristic function in E_m of radial symmetry about the origin. That we get all functions of the class $\mathfrak{P}(\mathfrak{H})$ as linear combinations of these particular functions, with non-negative coefficients, is stated by the following theorem.

THEOREM 2. *The class $\mathfrak{P}(\mathfrak{H})$ of functions which are positive definite in $\mathfrak{H}$ is identical with the class of functions of the form*

$$g(t) = \int_0^{\infty} e^{-t^2u^2}\,d\alpha(u), \tag{2.2}$$

where $\alpha(u)$ is non-decreasing and bounded for $u \geqq 0$.[11]

Indeed, since $\mathfrak{P}(\mathfrak{H})$ is convex and $\exp\{-t^2u^2\} \in \mathfrak{P}(\mathfrak{H})$, we see that $g(t)$, of (2.2), belongs to $\mathfrak{P}(\mathfrak{H})$. There remains to prove the converse. Now let $g(t)$ be an element of $\mathfrak{P}(\mathfrak{H})$. As $g(t) \in \mathfrak{P}(E_m)$, we know by Theorem 1 that it is of the form

$$g(t) = \int_0^{\infty} \Omega_m(tu)\,d\alpha_m(u), \qquad (m = 1, 2, 3, \cdots), \tag{2.3}$$

---

[11] Theorem 2 and Theorem 4 below (i.e. Theorem 1 of [16]) allow us to state the following result: *A separable semi-metric space $\mathfrak{S}$ is imbeddable in $\mathfrak{H}$ if and only if $\mathfrak{P}(\mathfrak{H}) \subset \mathfrak{P}(\mathfrak{S})$.* If we replace $\mathfrak{H}$ by E_m, then the theorem is no longer true without further restrictions on $\mathfrak{S}$. To show this, consider E_3, and let $\mathfrak{S}$ be a regular simplex σ_4 of E_4 of side 1, i.e., $\mathfrak{S}$ be the set of five points $P_0, \cdots, P_4$ with $P_iP_k = 1 (i \neq k)$. The statement $\mathfrak{P}(E_3) \subset \mathfrak{P}(\mathfrak{S})$ means (by Theorem 1 below for $m = 3$) that the family of functions $\Omega_3(tu)$ is positive definite in $\mathfrak{S} = \sigma_4$. As the $\Omega_3(tu)$ are known to be positive definite in E_3, our last statement amounts to the inequality

$$(*) \qquad \Delta(u) = \det \| \Omega_3(u \cdot P_iP_k) \|_{0,4} \geqq 0, \text{ for real } u.$$

Now $P_iP_k = \delta_{ik}$ implies $\Delta(u) = (1 + \Omega_3(u))^4(1 + 4\Omega_3(u))$. Hence (*) holds, since the inequality

$$(**) \qquad \Omega_3(u) = \frac{\sin u}{u} > -\tfrac{1}{4}$$

happens to be valid for all real u. Thus $\mathfrak{P}(E_3) \subset \mathfrak{P}(\sigma_4)$, without, of course, σ_4 being imbeddable in E_3. In order that $\mathfrak{P}(E_m) \subset \mathfrak{P}(\mathfrak{S})$ should imply that $\mathfrak{S}$ is imbeddable in E_m, it will probably be sufficient to assume that $\mathfrak{S}$ is made up of at least $m + 3$ points, an assumption suggested by certain results of Menger, [10].

$\{\alpha_m(u)\}$ being a certain sequence of uniformly bounded monotone functions. From the power series expansion (1.8) we derive the limiting relation

$$\lim_{m\to\infty} \Omega_m(r\sqrt{(2m)}) = e^{-r^2}. \tag{2.4}$$ [12]

As on the other hand (2.3) may be written in the form

$$g(t) = \int_0^\infty \Omega_m(tu\sqrt{(2m)})\, d\alpha_m(u\sqrt{(2m)}), \qquad (m = 1, 2, 3, \cdots), \tag{2.5}$$

we shall expect to be able to pass from (2.5) to the representation (2.2), of $g(t)$, by means of (2.4). Now it is easy to show that (2.4) holds uniformly within any finite interval of the r-axis (as well as in any finite domain of the complex r-plane). This, however, is insufficient to insure the applicability of Helly's well known convergence theorem for Stieltjes integrals, as our interval of integration is infinite. We shall have to show that (2.4) holds uniformly *for all real values of* r.[13]

2.2. The following proof of the uniformity of convergence in (2.4) is elementary.[14] For the sake of clarity we present it in the form of three lemmas.

LEMMA 1. *Let*

$$F(x) = \int_0^1 \cos(x\tau(u))\, du, \tag{2.6}$$

where $\tau(u)$ *is continuous for* $0 \leqq u \leqq 1$, $\tau(0) = 0$, *and* $\tau'(u)$ *non-decreasing for* $0 \leqq u < 1$, $\tau'(0) > k > 0$. *We have*

$$|F(x)| < \frac{1}{kx} \qquad \text{for } x > 0. \tag{2.7}$$

---

[12] The author is indebted to J. von Neumann for pointing out to him this relation.

[13] To stress the decisive rôle of the uniformity of convergence in (2.4) for all real r, we point out the following fact. The expansion (1.8) shows that

$$\lim_{m\to\infty} \Omega_m(r) = 1, \tag{2.4'}$$

uniformly in any finite r-interval. From this and (2.3) we may be tempted to expect, as $m \to \infty$, that

$$g(t) = \int_0^\infty 1 \cdot d\alpha(u),$$

and therefore conclude that $\mathfrak{P}(\mathfrak{H})$ is made up of constant functions only. This however, is not true, as e.g. $\exp\{-t^2\} \in \mathfrak{P}(\mathfrak{H})$. The point is that (2.4') does not hold uniformly for all real r.

[14] None of the available asymptotic estimates of Bessel functions was found to apply in the present situation.

Indeed, as $\tau'(u) > k$ holds throughout [0, 1], the second mean value theorem gives

$$F(1) = \int_0^1 \cos(\tau(u))\,du = \int_0^1 \frac{1}{\tau'(u)} \cos(\tau(u))\tau'(u)\,du$$

$$= \frac{1}{\tau'(0)} \int_0^\xi \cos(\tau(u))\tau'(u)\,du = \frac{1}{\tau'(0)} (\sin\tau(\xi) - \sin\tau(0)) = \frac{\sin\tau(\xi)}{\tau'(0)},$$

which implies

$$|F(1)| = \left| \int_0^1 \cos(\tau(u))\,du \right| < \frac{1}{k}.$$

Applying this inequality to $x\tau(u)$ instead of $\tau(u)$ (for a fixed positive value of x), and therefore with kx, instead of k, we immediately get (2.7).

LEMMA 2. *We have*

$$|\Omega_m(x\sqrt{(2m)})| < \frac{1}{x} \quad \text{for} \quad x > 0, \qquad m = 3, 4, 5, \cdots, \tag{2.8}$$

Indeed, passing to the variable $t = \cos\theta$ in

$$\Omega_m(x\sqrt{(2m)}) = \int_0^{\pi/2} \cos(x\sqrt{(2m)}\cos\theta) \sin^{m-2}\theta\,d\theta \Big/ \int_0^{\pi/2} \sin^{m-2}\theta\,d\theta,$$

we get

$$\Omega_m(x\sqrt{(2m)}) = \int_0^1 \cos(x\sqrt{(2m)}t)(1 - t^2)^{(m-3)/2}\,dt \Big/ \int_0^1 (1 - t^2)^{(m-3)/2}\,dt,$$

$$(m \geqq 3).$$

By means of the function

$$u = \int_0^t (1 - t^2)^{(m-3)/2}\,dt \Big/ \int_0^1 (1 - t^2)^{(m-3)/2}\,dt, \qquad (0 \leqq t \leqq 1), \tag{2.9}$$

whose inverse we denote by $t = t(u)(0 \leqq u \leqq 1)$, this goes over into

$$\Omega_m(x\sqrt{(2m)}) = \int_0^1 \cos(x\sqrt{(2m)}t(u))\,du.$$

With

$$\tau(u) = \sqrt{(2m)}t(u), \tag{2.10}$$

we finally have

$$\Omega_m(x\sqrt{(2m)}) = \int_0^1 \cos(x\tau(u))\,du. \tag{2.11}$$

The change of variable and the inversion $t = t(u)$, of (2.9) are justified by the fact that the function (2.9) is continuous and strictly increasing for $0 \leqq t \leqq 1$,

with $u(0) = 0$, $u(1) = 1$. Moreover

$$\tau'(u) = \sqrt{(2m)}t'(u) = \sqrt{(2m)}(1 - t^2)^{-(m-3)/2} \cdot \int_0^1 (1 - t^2)^{(m-3)/2} dt, \qquad (0 \leqq t < 1),$$

is an increasing function and

$$\tau'(0) = \sqrt{(2m)} \int_0^1 (1 - t^2)^{(m-3)/2} dt$$

$$= \sqrt{(2m)} \int_0^{\pi/2} \sin^{m-2} \theta \, d\theta \geqq \sqrt{(2m)} \frac{(m-3)!!}{(m-2)!!} > 1.^{15} \tag{2.12}$$

Now (2.11), (2.12) and Lemma 1 imply the required inequality (2.8).

LEMMA 3. *The limiting relation*

$$\lim_{m \to \infty} \Omega_m(x\sqrt{(2m)}) = e^{-x^2} \tag{2.13}$$

holds uniformly for $-\infty < x < \infty$.

Indeed, the difference

$$| \Omega_m(x\sqrt{(2m)}) - e^{-x^2} |$$

will be $< \epsilon$ for real x with $| x | > A(\epsilon)$, and $m \geqq 3$, since it falls below $x^{-1} + e^{-x^2}$ by Lemma 2. Now for sufficiently large m it will likewise be $< \epsilon$ for $| x | \leqq A(\epsilon)$, since (2.13) holds uniformly in every finite interval as already mentioned.

2.3. We can now readily complete a proof of Theorem 2. Returning to the relation (2.5), Helly's compactness theorem insures the existence of a subsequence $\{\alpha_{m_\nu}(u\sqrt{(2m_\nu)})\}$ converging to the monotone function $\alpha(u)(\alpha(0) = 0)$ in all points of continuity of $\alpha(u)$ ([8], p. 286). Let t assume a fixed positive value. By Lemma 3 we know that $\Omega_{m_\nu}(tu\sqrt{(2m_\nu)})$ tends to $\exp\{-t^2u^2\}$ uniformly for all real values of u, hence (2.5) implies

$$g(t) = \int_0^\infty e^{-t^2u^2} d\alpha_{m_\nu}(u\sqrt{(2m_\nu)}) + \sigma,$$

---

[15] Here we write $k!! = k(k-2)(k-4)\cdots$. The last inequality (2.12) can be verified by Stirling's formula. I owe the following interesting direct proof to F. Bohnenblust. Let $m = 2n + 2$ be even. Then

$$\frac{(2n-1)!!}{(2n)!!}\sqrt{(4n+4)}$$

$$= \frac{(2n-1)(2n-3)\cdots 3\cdot 1}{\sqrt{(2n)}(\sqrt{(2n)}\sqrt{(2n-2)})(\sqrt{(2n-2)}\sqrt{(2n-4)})\cdots(\sqrt{4}\sqrt{2})\sqrt{2}}\sqrt{(4n+4)}$$

$$\geqq \frac{(2n-1)(2n-3)\cdots 3\cdot 1}{\sqrt{(2n)}(2n-1)(2n-3)\cdots 3\cdot\sqrt{2}}\sqrt{(4n+4)} = \sqrt{\left(\frac{n+1}{n}\right)} > 1.$$

since $\sqrt{(2n(2n-2))} < 2n - 1$. Similar argument in case m is odd.

with $|\sigma| < \epsilon$ for sufficiently large values of ν. But this implies that

$$g(t) = \int_0^l e^{-t^2u^2}\, d\alpha_{m_\nu}(u\sqrt{(2m_\nu)}) + \rho,$$

with $|\rho| < 2\epsilon$, provided both l and ν are sufficiently large. If $u = l$ is a continuity point of $\alpha(u)$, letting $\nu \to \infty$, we get ([8], pp. 288–289)

$$g(t) = \int_0^l e^{-t^2u^2}\, d\alpha(u) + \rho_1. \qquad |\rho_1| \leqq 2\epsilon.$$

Now letting $l \to \infty$ and then $\epsilon \to 0$ we get the desired representation (2.2). This formula, established for $t > 0$, holds also if $t = 0$ in view of the continuity of its two sides.

3. Completely monotone functions

3.1. A real function $f(t)$ is said to be *completely monotone for* $t \geqq 0$, if

$$(-1)^n f^{(n)}(t) \geqq 0 \quad \text{for} \quad 0 < t < \infty, \qquad (n = 0, 1, 2, \cdots), \tag{3.1}$$

and

$$f(0) = f(+0), \tag{3.1'}$$

the last condition expressing the continuity of $f(t)$ at the origin. A fundamental theorem[16] states the identity of this class with the class of functions representable as a Laplace-Stieltjes integral

$$f(t) = \int_0^\infty e^{-tu}\, d\beta(u), \tag{3.2}$$

where $\beta(u)$ is non-decreasing and bounded for $u \geqq 0$. *We shall denote this class of functions throughout this paper by the symbol* $\mathfrak{M}$.

If $g(t)$ is positive definite in $\mathfrak{H}$, then we may write in view of Theorem 2

$$g(t) = \int_0^\infty e^{-t^2u^2}\, d\alpha(u) = \int_0^\infty e^{-t^2u}\, d\alpha(\sqrt{u}) = \int_0^\infty e^{-t^2u}\, d\beta(u) = f(t^2).$$

Thus Theorem 2 and the Laplace integral representation (3.2) immediately prove the following theorem.

THEOREM 3. *A function $f(t)$ is completely monotone for $t \geqq 0$, if and only if $f(t^2)$ is positive definite in Hilbert space $\mathfrak{H}$. In symbols: $f(t) \in \mathfrak{M}$ if and only if $f(t^2) \in \mathfrak{P}(\mathfrak{H})$.*

The elements $g(t)$ of $\mathfrak{P}(\mathfrak{H})$ and $f(t)$ of $\mathfrak{M}$ are therefore in a one-to-one correspondence expressed by the relation $g(t) = f(t^2)$, which is valid both ways.

Definition 1 of positive definite functions and Theorem 3 allow us to state that the conditions (3.1) and (3.1′) for complete monotoneity of a function $f(t)$ are entirely equivalent with the following two requirements:

1. *$f(t)$ be real and continuous for $t \geqq 0$.*

[16] See [7], [1], [19], and [13].

2. *If $P_1, P_2, \cdots, P_n$ are any n points of a euclidean space ($n = 1, 2, 3, \cdots$), we should have*

$$\begin{vmatrix} f(P_1P_1^2) & f(P_1P_2^2) & \cdots & f(P_1P_n^2) \\ f(P_2P_1^2) & f(P_2P_2^2) & \cdots & f(P_2P_n^2) \\ \vdots & \vdots & & \vdots \\ f(P_nP_1^2) & f(P_nP_2^2) & \cdots & f(P_nP_n^2) \end{vmatrix} \geqq 0.^{17} \tag{3.3}$$

A direct derivation of (3.3) as a consequence of (3.1) and (3.1′) would furnish a new proof of the theorem that (3.1) and (3.1′) imply the representation (3.2).

The connection between the class $\mathfrak{P}$ of characteristic functions (0.2) and the class $\mathfrak{M}$ is now clear. We have the infinite sequence of classes

$$\mathfrak{P} \supset \mathfrak{P}(E_1) \supset \mathfrak{P}(E_2) \supset \cdots \supset \mathfrak{P}(E_m) \supset \cdots \supset \mathfrak{P}(\mathfrak{H}). \tag{3.4}$$

The class $\mathfrak{P}(E_1)$ is made up of those elements of $\mathfrak{P}$ which are of radial symmetry in E_1, i.e., are *even*, or merely *real*. Their form is obviously

$$g(t) = \int_0^\infty \cos tu \, d\alpha(u),$$

which is in accord with Theorem 1, since $\Omega_1(r) = \cos r$. A function $g(t)$ of $\mathfrak{P}(E_1)$ belongs to $\mathfrak{P}(E_m)$ if and only if

$$g(\sqrt{(x_1^2 + \cdots + x_m^2)}) \text{ is a characteristic function of } E_m. \tag{3.5}$$

The completely monotone $f(t^2)(f(t) \in \mathfrak{M})$ are identical with those elements $g(t)$ of $\mathfrak{P}(E_1)$ which enjoy the property (3.5) for *all* values of m.

3.2. Further light is thrown on the gradual narrowing down of the classes (3.4) by the requirement of positive definiteness in E_m, for increasing m, by the following lemma.

LEMMA 4. *The functions*

$$g(t) = \int_0^\infty \Omega_m(tu) \, d\alpha(u) \tag{3.6}$$

of the class $\mathfrak{P}(E_m)$ are $\left[\frac{m-1}{2}\right]$-times differentiable.

This fact shows clearly the smoothing effect on $g(t)$ of the requirement of positive definiteness in higher euclidean spaces. It implies in particular that the elements of $\mathfrak{P}(\mathfrak{H})$, and those of $\mathfrak{M}$, are ∞-often differentiable. Theorem 2, moreover, shows that the elements $g(t)$ of $\mathfrak{P}(\mathfrak{H})$ are even analytic in the sector $|\arg t| < \pi/4$ of the complex t-plane.

In order to prove Lemma 4 it suffices to consider

$$g^{(\nu)}(t) = \int_0^\infty \frac{\partial^\nu}{\partial t^\nu} \Omega_m(tu) \, d\alpha(u) \tag{3.7}$$

[17] See Theorem 3′ below which excludes the equality sign in (3.3), except in trivial cases.

and to see for what values of $\nu = 1, 2, 3, \cdots$ this integral converges for all values of t. The integrand is

$$\frac{\partial^\nu}{\partial t^\nu}\Omega_m(tu) = \Omega_m^{(\nu)}(tu)\cdot u^\nu. \tag{3.8}$$

From $\Omega_m(r) = c_m J_k(r) r^{-k}$, where $k = (m-2)/2$ and c_m is a positive constant, we get by the rule of Leibnitz

$$c_m^{-1}\Omega_m^{(\nu)}(r) = J_k^{(\nu)}(r)r^{-k} - \binom{\nu}{1} k J_k^{(\nu-1)}(r) r^{-k-1} + \cdots \pm k(k+1)\cdots(k+\nu-1)J_k(r)r^{-k-\nu}.$$

As $J_k^{(\nu)}(r) = O(r^{-\frac{1}{2}})$ we have

$$\Omega_m^{(\nu)}(r) = O(r^{-k-\frac{1}{2}}) = O(r^{-\frac{1}{2}(m-1)}),$$

and hence by (3.8)

$$\frac{\partial^\nu}{\partial t^\nu}\Omega_m(tu) = O(u^{\nu-\frac{1}{2}(m-1)}).$$

Thus (3.7) converges absolutely as long as $\nu \leqq (m-1)/2$, proving our lemma.

In view of Lemma 4 there are strong reasons to believe that the continuity assumption on $g(t)$ in Definition 1, section 1.1, may be dropped as soon as $\mathfrak{S}$ is an euclidean space E_m of dimension m exceeding one. More precisely: The inequality (1.2″) should imply the continuity of $g(t)$ everywhere, with the exception of the origin $t = 0$. The last discontinuity must obviously remain, for if $g(t)$ satisfies the inequalities (1.2″), it will still satisfy them if the value of $g(0)$ is increased.

3.3. We close this section with the following theorem.

THEOREM 3′. *If $P_1, P_2, \cdots, P_n$ are any n distinct points of E_{n-1} and $f(t) \in \mathfrak{M}$, then*

$$\det \| f(P_j P_k^2) \|_{1,n} > 0, \tag{3.9}$$

unless $f(t)$ reduces to a non-negative constant.

Indeed, let us show first that the quadratic form $\sum_{j,k=1}^{n} \exp\{-P_j P_k^2\}\rho_j\rho_k$ is positive definite. Transforming the integral (2.1) ($u = 1$) to polar coördinates and using again vector notation in $E_m (m = n-1)$, with $P_j = x^j = (x_1^j, \cdots, x_m^j)$, we get

$$e^{-|x|^2} = 2^{1-m}[\Gamma(m/2)]^{-1}\int_0^\infty M_\xi\{e^{i(x\xi)r}\}e^{-r^2/4}r^{m-1}\,dr.$$

Replacing the vector $x = x - 0$ by $P_j - P_k = x^j - x^k$, we get

$$e^{-P_j P_k^2} = c\cdot\int_0^\infty M_\xi\{e^{i(x^j\xi)r}\cdot e^{-i(x^k\xi)r}\}e^{-r^2/4}r^{m-1}\,dr,$$

Now

$$\sum_{j,k=1}^{n} e^{-P_jP_k^2}\rho_j\rho_k = c\cdot\int_0^\infty M_\xi\left\{\left|\sum_{j=1}^{n} e^{i(x^j\xi)r}\cdot\rho_j\right|^2\right\}e^{-r^2/4}r^{m-1}\,dr,$$

which proves the positive definite character of this form by an elementary argument previously used (see [15], p. 790). Hence

$$\sum_{j,k=1}^{n} e^{-u\cdot P_jP_k^2}\rho_j\rho_k > 0, \qquad (u > 0,\ \rho_1^2 + \cdots + \rho_n^2 > 0). \tag{3.10}$$

Now (3.2) and (3.10) show that

$$\sum_{1}^{n} f(P_jP_k^2)\rho_j\rho_k = \int_0^\infty \left(\sum_{1}^{n} e^{-uP_jP_k^2}\rho_j\rho_k\right)d\beta(u)$$

is positive, if $\rho_1^2 + \cdots + \rho_n^2 > 0$, unless $\beta(u)$ reduces to a step function with a single saltus at the origin while $\sum \rho_j\rho_k = (\rho_1 + \cdots + \rho_n)^2 = 0$. But then $f(t)$ is a constant as was to be proved.

4. Integrals of completely monotone functions

4.1. We have denoted by $\mathfrak{M}$ the class of functions $f(t)$ completely monotone for $t \geqq 0$. If we drop the second condition (3.1′), which requires continuity at the origin, we get a larger class of functions $\psi(t)$ defined for $t > 0$ and satisfying the inequalities

$$(-1)^n\psi^{(n)}(t) \geqq 0 \quad \text{for} \quad 0 < t < \infty, \qquad (n = 0, 1, 2, \cdots). \tag{4.1}$$

These functions are also representable as Laplace integrals

$$\psi(t) = \int_0^\infty e^{-tu}\,d\gamma(u) \quad \text{for} \quad 0 < t < \infty, \tag{4.2}$$

where $\gamma(u)$ is non-decreasing. $\psi(0) = \psi(+0)$ exists if and only if $\gamma(u)$ is bounded, in which case $\psi(0) = \psi(+0) = \gamma(\infty)$. Integrating (4.2) between ϵ and $t(0 < \epsilon < t)$ we get

$$\int_\epsilon^t \psi(t)\,dt = \int_0^\infty \frac{e^{-u\epsilon} - e^{-ut}}{u}\,d\gamma(u). \tag{4.3}$$

Letting here $\epsilon \to 0$ we get

$$\phi(t) = \int_{+0}^{t}\psi(t)\,dt = \int_0^\infty \frac{1 - e^{-ut}}{u}\,d\gamma(u), \qquad (0 < t < \infty), \tag{4.4}$$

where both improper integrals converge or diverge simultaneously. We have convergence for all $t > 0$, if and only if

$$\int_1^\infty \frac{d\gamma(u)}{u} \tag{4.5}$$

exists.

DEFINITION 2. *We denote by T the class of functions $\phi(t)$ defined by* (4.4), *and* $\phi(0) = 0$, *as integrals of those functions $\psi(t)$ completely monotone for $t > 0$ for which the improper integral $\int_{+0}\psi(t)dt$ converges. In terms of the Laplace representation* (4.2), *this last mentioned condition is equivalent with the assumption* (4.5).

This class T is fundamental in our subsequent work. In this section we only prove a certain closure property of this class expressed by the following lemma.

LEMMA 5. *The class T is closed with respect to ordinary convergence to a continuous limit function, that is, if $\{\phi_n(t)\}$ $(n = 1, 2, 3, \cdots)$ is a sequence of elements of T and $\phi_0(t)$ is continuous for $t \geqq 0$, then*

$$\lim_{n\to\infty} \phi_n(t) = \phi_0(t), \qquad (0 \leqq t < \infty), \tag{4.6}$$

implies that $\phi_0(t) \in T$.

Indeed, by assumption we have

$$\phi_n(t) = \int_{+0}^{t} \psi_n(t)\, dt, \qquad (n = 1, 2, 3, \cdots), \tag{4.7}$$

where $\psi_n(t)$ are completely monotone for $t > 0$ and all improper integrals converge. Now the $\psi_n(t)$ are analytic in the half-plane $\Re t > 0$ and therefore also their integrals $\phi_n(t)$ are of this nature. Let $t = \sigma + i\tau$. We want to show that the functions of the sequence $\{\phi_n(t)\}$ are uniformly bounded in the rectangular domain

$$D\colon\ \sigma_1 \leqq \sigma \leqq \sigma_2\,, \qquad |\tau| \leqq \tau_1\,, \qquad (0 < \sigma_1 < \sigma_2).$$

Indeed

$$\phi_n(t) = \phi_n(\sigma + i\tau) = \int_0^\infty (1 - e^{-\sigma u - iu\tau})u^{-1}\, d\gamma_n(u)$$

$$= \int_0^\infty (1 - e^{-\sigma u})u^{-1}\, d\gamma_n(u) + \int_0^\infty e^{-\sigma u}(1 - e^{-iu\tau})u^{-1}\, d\gamma_n(u).$$

Hence

$$|\phi_n(t)| \leqq \phi_n(\sigma) + |\tau| \int_0^\infty e^{-\sigma u} \left|\sin \frac{u\tau}{2} \Big/ \frac{u\tau}{2}\right| d\gamma_n(u) \leqq \phi_n(\sigma) + \tau_1 \int_0^\infty e^{-\sigma u}\, d\gamma_n(u)$$

$$\leqq \phi_n(\sigma_2) + \tau_1 \int_0^\infty e^{-\sigma_1 u}\, d\gamma_n(u) \leqq \phi_n(\sigma_2) + \tau_1 \int_0^\infty (1 - e^{-\sigma_1 u})(\sigma_1 u)^{-1} d\gamma_n(u),$$

since $e^{-x} < (1 - e^{-x})x^{-1} (x > 0)$. We therefore have

$$|\phi_n(t)| \leqq \phi_n(\sigma_2) + \tau_1\sigma_1^{-1}\phi_n(\sigma_1) \qquad (t \text{ in } D)$$

This proves, in view of (4.6), that the $\phi_n(t)$ are uniformly bounded in D. By Vitali's convergence theorem we conclude that (4.6) holds uniformly in any bounded domain inside the half-plane $\Re t > 0$ and that $\phi_0(t)$ is analytic. (See Titchmarsh, [17], p. 168.)

Let $\psi_0(t) = \phi_0'(t)$; now (4.6) implies

$$\lim_{n\to\infty} (-1)^k \psi_n^{(k)}(t) = (-1)^k \psi_0^{(k)}(t), \qquad (k = 0, 1, 2, \cdots),$$

if $\Re t > 0$, hence in particular for all positive values of t. Thus $(-1)^k \psi_n^{(k)}(t) \geqq 0$ implies $(-1)^k \psi_0^{(k)}(t) \geqq 0$ $(t > 0;\ k = 0, 1, 2, \cdots)$, hence $\psi_0(t)$ is completely monotone for $t > 0$. The proof is now immediately completed, for

$$\phi_0(t) - \phi_0(\epsilon) = \int_\epsilon^t \psi_0(t)\, dt, \quad (0 < \epsilon < t), \quad \text{and} \quad \phi_0(+0) = \phi_0(0) = 0,$$

imply, as $\epsilon \to 0$, the existence of $\int_{+0}\psi_0(t)dt$ and the relation

$$\phi_0(t) = \int_{+0}^t \psi_0(t)\, dt, \qquad (0 < t < \infty).$$

Hence $\phi_0(t) \in T$.

We conclude this section with two remarks concerning Lemma 5. The function theoretic argument shows that if 1. $\phi_n(t)$ are uniformly bounded in any finite interval $0 \leqq t \leqq \sigma_2$, 2. the limit (4.6) exists in a set of positive values of t with a positive limit point, then (4.6) holds for all t with positive real part, and $\phi_0(t)$ is analytic for $\Re t > 0$. In concluding, however, that $\phi_0(t) \in T$, it is essential to assume that $\phi_0(+0) = 0$, i.e., with $\phi_0(0) = 0$, $\phi_0(t)$ should be continuous at the origin. This fact was used in the last step of our proof. That $\phi_0(+0) = 0$ is not implied by (4.6), is seen by the following sequence: $\phi_n(t) = 1 - e^{-nt}$, $(\psi_n(t) = ne^{-nt})$, for which (4.6) holds for $t \geqq 0$ with $\phi_0(t) = 1(t > 0)$, $\phi_0(0) = 0$. Our second remark is that a somewhat more involved but elementary proof of Lemma 5 (free of any appeal to analytic function theory) is available.

II. Determination of All Metric Transforms of Hilbert Space Which Are Isometrically Imbeddable in Hilbert Space. Applications to Completely Monotone Functions

5. Isometric imbedding in $\mathfrak{H}$ and positive definite functions

5.1. Before we enter into a discussion of our specific topic, a few general remarks concerning metric transforms are necessary. Let $\mathfrak{S}$ be a semi-metric space with the metric PP'. In a study of its metric transforms $F(\mathfrak{S})$ by a transforming function $F(t)$, the following distinction is of interest. We shall say that $F(t)$ is a *proper* transforming function if $F(t) > 0$ for all positive values of t; if, however, $F(t)$ has positive zeros, we call $F(t)$ an *improper* transforming function. The reason for this distinction is as follows. If $F(t)$ is proper, then $F(\mathfrak{S})$ is obviously semi-metric. This, however, is not always the case if $F(t)$ is improper, for the only properties of the metric $F(PP')$ of $F(\mathfrak{S})$ are 1. $F(PP') = F(P'P) \geqq 0$, 2. $F(PP) = 0$.[18] Operating with $\mathfrak{S}$, rather than $F(\mathfrak{S})$, we are

[18] Of course *some* transforms $F(\mathfrak{S})$ may still be semi-metric. Thus if $F(t) = |\sin t|$ $(t \geqq 0)$, we see that $F(\mathfrak{S})$ is semi-metric if the diameter of $\mathfrak{S}$ is $< \pi$.

thus led to consider spaces $\mathfrak{S}$ whose distance function PP' enjoys only the following weaker properties

$$1.\ PP' = P'P \geqq 0, \qquad 2.\ PP = 0. \tag{5.1}$$

We shall refer to such spaces as *quasi-metric* spaces.

A quasi-metric space $\mathfrak{S}$ can not, in general, be turned into a semi-metric space by the identification of points P, Q for which $PQ = 0$ (similar to the identification of measurable functions equal almost everywhere), for this identification does not enjoy the necessary transitive property. Indeed, if $PQ = 0$ and $QR = 0$, there is no reason why $PR = 0$ should hold. Moreover, even if this were always true, then $PQ = 0$ need not imply $PS = QS$, where S is a third point of $\mathfrak{S}$. It becomes now apparent that this identification is possible and turns $\mathfrak{S}$ into a semi-metric space $\mathfrak{R}$, if and only if $\mathfrak{S}$ is isometrically imbeddable in a semi-metric space $\mathfrak{R}$.

For the sake of generality we include also *improper* transforming functions $F(t)$ in our discussion, even $F(t) \equiv 0$ not being excluded. Accordingly we are bound to deal with *quasi-metric* spaces and their imbedding in $\mathfrak{H}$.

5.2. Our imbedding problem $\{E_m\ ;\ \mathfrak{H}\}$ (see Introduction) is of the following general type: *Given a separable semi-metric space $\mathfrak{S}$, determine all functions $F(t)$, proper or improper, such that the metric transform $F(\mathfrak{S})$ be imbeddable in $\mathfrak{H}$. We shall denote by $\Pi(\mathfrak{S})$ the class of functions $F(t)$ which are solutions of this problem $\{\mathfrak{S};\ \mathfrak{H}\}$.* In particular $\Pi(E_m)$ and $\Pi(\mathfrak{H})$ represent the totality of solutions of the problems $\{E_m\ ;\ \mathfrak{H}\}$ and $\{\mathfrak{H}\ ;\ \mathfrak{H}\}$, respectively.

A characteristic property of a function $F(t)$ in $\Pi(\mathfrak{S})$ is as follows. If $P_0, P_1, \cdots, P_n$ are any $n+1$ points of $\mathfrak{S}$, then the distances $F(P_iP_k)$ are the edges of a euclidean simplex, i.e., there exist $n+1$ points $Q_0, \cdots, Q_n$ in E_n, such that $F(P_iP_k)$ = the euclidean distance between Q_i and Q_k. As $F(t)$ thus turns every finite set of points of $\mathfrak{S}$ into a euclidean simplex, $F(t)$ may be called a *simplicial function* of $\mathfrak{S}$[19], and the whole class $\mathfrak{P}(\mathfrak{S})$ may be referred to as the *simplicial class* of $\mathfrak{S}$. Note that $\Pi(\mathfrak{S})$ is never empty, for $F(t) \in \Pi(\mathfrak{S})$, if $F(t) \equiv 0$.

5.3. A connecting link between problems of isometric imbedding in $\mathfrak{H}$ and the theory of positive definite functions is furnished by the following theorem.

THEOREM 4. *A necessary and sufficient condition that a separable quasi-metric space $\mathfrak{S}$ be isometrically imbeddable in $\mathfrak{H}$ is that the family of functions* $\exp\{-\lambda t^2\}$ *(where λ is a positive parameter) be positive definite in $\mathfrak{S}$.*[20]

[19] The term "simplicial function" was suggested to the author by L. M. Blumenthal. Blumenthal had previously called $F(t)$ a "tetrahedral function" of $\mathfrak{S}$, if $F(\mathfrak{S})$ enjoys the 4-point property. His theorem mentioned in the Introduction may be stated as follows: $F(t) = t^\kappa (0 < \kappa \leqq \frac{1}{2})$ is a tetrahedral function of any *metric* space $\mathfrak{S}$. See [3].

[20] Definition 1, §1.1, of functions $g(t)$ which are positive definite in a semi-metric space $\mathfrak{S}$, applies unchanged in case $\mathfrak{S}$ is only quasi-metric.

A proof of this theorem is found in [16], §3. Let now $\mathfrak{S}$ be semi-metric and let $\mathfrak{P}(\mathfrak{S})$ be its positive definite class and $\Pi(\mathfrak{S})$ its simplicial class. An immediate consequence of Theorem 4 is the following characterization of the class $\Pi(\mathfrak{S})$ in terms of the class $\mathfrak{P}(\mathfrak{S})$.

THEOREM 5. *A function $F(t)(F(0) = 0)$ belongs to the simplicial class $\Pi(\mathfrak{S})$ of a semi-metric space $\mathfrak{S}$, if and only if the family of functions $\exp\{-\lambda F^2(t)\}$ $(\lambda > 0)$ belongs to the positive definite class $\mathfrak{P}(\mathfrak{S})$.*

Indeed, $F(t) \in \Pi(\mathfrak{S})$ is equivalent with the statement that $F(\mathfrak{S})$ is imbeddable in $\mathfrak{H}$. By Theorem 4 this is the case if and only if $\exp\{-\lambda t^2\}$ is positive definite in $F(\mathfrak{S})$, for all $\lambda > 0$, or, which is the same thing, if $\exp\{-\lambda F^2(t)\}$ is positive definite in $\mathfrak{S}$. This proves our theorem.

This general theorem will be used in this paper only in case $\mathfrak{S}$ is either a euclidean space or Hilbert space. For convenience we restate it in each of these cases as special corollaries.

COROLLARY 1. *$F(t) \in \Pi(E_m)$ if and only if $\exp\{-\lambda F^2(t)\} \in \mathfrak{P}(E_m)$ for all $\lambda > 0$.*

COROLLARY 2. *$F(t) \in \Pi(\mathfrak{H})$ if and only if $\exp\{-\lambda F^2(t)\} \in \mathfrak{P}(\mathfrak{H})$ for all $\lambda > 0$.*

Corollary 1 was already discussed, in case $m = 1$, in [16], §4, in connection with the class $\Pi(E_1)$ of "screw functions" of $\mathfrak{H}$; here it will be applied in §8 of the Appendix.

5.4. We shall presently use Corollary 2 in establishing the following theorem which determines the class $\Pi(\mathfrak{H})$, i.e., the totality of solutions of the problem $\{\mathfrak{H}\,;\,\mathfrak{H}\}$.

THEOREM 6. *The simplicial class $\Pi(\mathfrak{H})$, that is, the totality of continuous functions $F(t)$ such that the metric transform $F(\mathfrak{H})$ be isometrically imbeddable in $\mathfrak{H}$, is identical with the class of functions of the form*

$$F(t) = \left\{ \int_0^\infty \frac{1 - e^{-t^2 u}}{u}\, d\gamma(u) \right\}^{1/2}, \qquad (t \geqq 0), \tag{5.2}$$

where $\gamma(u)$ is non-decreasing for $u \geqq 0$ and such that

$$\int_1^\infty \frac{d\gamma(u)}{u} \text{ exists.} \tag{5.3}$$

The proof that (5.2) furnishes functions $F(t)$ belonging to $\Pi(\mathfrak{H})$ is very simple (see [16], 2nd footnote of §3). For let $P_0, \cdots, P_n$ be any $n + 1$ points of $\mathfrak{H}$. In order to prove that $F(t) \in \Pi(\mathfrak{H})$, i.e., $F(\mathfrak{H})$ is imbeddable in $\mathfrak{H}$, it suffices to show that $F(P_iP_k)$ are distances of $n + 1$ points of E_n. As $\exp\{-ut^2\}(u \geqq 0)$ belongs to $\mathfrak{P}(E_m)$, we have

$$\sum_0^n e^{-uP_iP_k^2} \rho_i \rho_k \geqq 0 \qquad \text{for} \qquad u \geqq 0. \tag{5.4}$$

Hence if

$$\sum_0^n \rho_i = 0, \tag{5.5}$$

we conclude from (5.2), (5.4) and (5.5) that

$$\sum_0^n F^2(P_iP_k)\rho_i\rho_k = -\int_0^\infty \left(\sum_0^n e^{-uP_iP_k^2}\rho_i\rho_k\right) u^{-1}\, d\gamma(u) \leqq 0\,. \tag{5.6}$$

The fact that $\sum F^2(P_iP_k)\rho_i\rho_k \leqq 0$ for any set of real ρ_i subject to the relation (5.5), is equivalent to the statement that $F(P_iP_k)$ are distances of $n+1$ points of E_n.[21] This completes the first part of the proof.

Let us now prove the converse to the effect that an element $F(t)$ of $\Pi(\mathfrak{H})$ is necessarily of the form (5.2). From Corollary 2 we learn that $\exp\{-\lambda F^2(t)\}$ is positive definite in $\mathfrak{H}$ for all positive values of λ. By Theorem 2 it is therefore of the form

$$e^{-\lambda F^2(t)} = \int_0^\infty e^{-t^2u}\, d\alpha(u,\lambda)\,, \qquad (\lambda > 0), \tag{5.7}$$

where $\alpha(u,\lambda)$ is a family of non-decreasing functions of $u \geqq 0$, defined for all $\lambda > 0$, with

$$\alpha(0,\lambda) = 0, \qquad \alpha(\infty,\lambda) = 1. \tag{5.8}$$

For convenience we introduce the function

$$\phi(t) = F^2(\sqrt{t}) \tag{5.9}$$

in terms of which (5.7) becomes

$$e^{-\lambda\phi(t)} = \int_0^\infty e^{-tu}\, d\alpha(u,\lambda)\,, \qquad (\lambda > 0). \tag{5.10}$$

In view of (5.2) and (5.9) our objective is to show that $\phi(t)$ is of the form

$$\phi(t) = \int_0^\infty \frac{1-e^{-tu}}{u}\, d\gamma(u)\,, \tag{5.11}$$

with $\gamma(u)$ subject to the restriction (5.3). In different words: That $\phi(t)$ belongs to the class T of §4. This fact is now readily established. Indeed (5.10) and the relation $1 = \int_0^\infty d\alpha(u,\lambda)$ imply

$$\frac{1-e^{-\lambda\phi(t)}}{\lambda} = \int_0^\infty \frac{1-e^{-tu}}{\lambda}\, d\alpha(u,\lambda) = \int_0^\infty \frac{1-e^{-tu}}{u}\cdot\frac{u}{\lambda}\, d\alpha(u,\lambda)\,,$$

whence

$$\frac{1-e^{-\lambda\phi(t)}}{\lambda} = \int_0^\infty \frac{1-e^{-tu}}{u}\, d\beta(u,\lambda)\,, \qquad (t \geqq 0, \lambda > 0), \tag{5.12}$$

[21] Here we use the following theorems: 1. *The non-negative quantities a_{ik} ($a_{ik} = a_{ki}$; $i,k = 0, 1, \cdots n$; $a_{ii} = 0$) are distances of $n+1$ points of E_n, if and only if* (5.5) *implies* $\sum_0^n a_{ik}^2\rho_i\rho_k \leqq 0$. (See [16], §3). 2. *A separable quasi-metric space is imbeddable in $\mathfrak{H}$, if and only if any $n+1$ of its points ($n = 2, 3, \cdots$) are congruent with $n+1$ points of $\mathfrak{H}$.* (See Menger, [9].)

where the new family of monotone functions $\beta(u, \lambda)$ is defined by

$$\beta(u, \lambda) = \frac{1}{\lambda} \int_0^u v \, d\alpha(v, \lambda), \qquad (\lambda > 0). \tag{5.13}$$

From (5.12) we see that

$$(1 - e^{-\lambda\phi(t)})\lambda^{-1}$$

belongs to the class T for all $\lambda > 0$. Letting $\lambda \to 0$ we find this function converging to the continuous limit function $\phi(t)$. On the basis of the closure property expressed by Lemma 5 (§4) we conclude that $\phi(t)$ belongs to T. This completes the proof of Theorem 6.

What has just been established may also be stated as follows: *There is a one-to-one correspondence between the classes T and $\Pi(\mathfrak{H})$ expressed by the relation*

$$\phi(t) = F^2(\sqrt{t}) \quad \text{or} \quad F(t) = \sqrt{\phi(t^2)}, \qquad (\phi \in T, F \in \Pi(\mathfrak{H})). \tag{5.14}$$

It is likewise interesting to connect $F(t)$ directly with the completely monotone functions. From (5.14) and (4.4) we get

$$\frac{d}{dt}\phi(t) = \frac{d}{dt}F^2(\sqrt{t}) = \psi(t). \qquad (t > 0).$$

We may therefore state the following theorem.

THEOREM 6′. *A necessary and sufficient condition that the non-negative continuous function $F(t)$, vanishing at the origin, have the property that $F(\mathfrak{H})$ be isometrically imbeddable in $\mathfrak{H}$, is that*

$$\frac{d}{dt}F^2(\sqrt{t}) = \psi(t) \tag{5.15}$$

be completely monotone for $t > 0$.

5.5. Let us apply the criterion expressed by Theorem 6′ to a few examples. Let $F_1(t) = t^\kappa (\kappa > 0)$. By (5.15) we get $\psi(t) = \kappa t^{\kappa-1} = \kappa/t^{1-\kappa}$ and this function is completely monotone if $1 - \kappa \geqq 0$, or $\kappa \leqq 1$. Hence

$$F_1(t) = t^\kappa, \qquad (0 < \kappa \leqq 1). \tag{5.16}$$

belongs to $\Pi(\mathfrak{H})$. Let now

$$F_2(t) = \frac{t}{1+t}, \tag{5.17}$$

which was mentioned in the Introduction. In this case $\psi(t) = (1 + \sqrt{t})^{-3}$ is likewise completely monotone. This fact may be verified directly by successive differentiations or more elegantly as follows: $(1 + t)^{-1}$ is obviously in $\mathfrak{M}$ and therefore also its cube $(1 + t)^{-3}$. Finally it will follow from the results of §6 (Theorem 8) that if $f(t)$ is in $\mathfrak{M}$, then so is $f(\sqrt{t})$. Our last two examples are the functions

$$F_3(t) = (1 - e^{-t^2})^{\frac{1}{2}}, \qquad F_4(t) = (\log(1 + t^2))^{\frac{1}{2}}. \tag{5.18}$$

We find that $\psi_3(t) = e^{-t}$, $\psi_4(t) = (1 + t)^{-1}$ are both in $\mathfrak{M}$, hence F_3, F_4 are elements of $\Pi(\mathfrak{H})$.

A glance at either (5.2) or (5.15) will show that, if we exclude from consideration the trivial function $F(t) \equiv 0$, all elements of $\Pi(\mathfrak{H})$ are *proper* transforming functions, that is, $F(t) > 0$ for $t > 0$. Moreover, if $F(t)$ is not of the form $F(t) = ct$ $(c \geqq 0)$, we actually have

$$\begin{gathered} F^2(\sqrt{t}) > 0, \qquad \frac{d}{dt} F^2(\sqrt{t}) > 0, \\ \frac{d^2}{dt^2} F^2(\sqrt{t}) < 0, \qquad \frac{d^3}{dt^3} F^2(\sqrt{t}) > 0, \cdots \qquad (t > 0). \end{gathered} \tag{5.19}$$

Hence: *The continuity of $F(t)$ for $t \geqq 0$, $F(0) = 0$, and the inequalities* (5.19) *characterize completely all elements of $\Pi(\mathfrak{H})$, with the exception of the elements $F(t) = ct$ $(c \geqq 0)$.*

Ruling out the case $F(t) \equiv 0$, we see that $\mathfrak{H}$ and $F(\mathfrak{H})$ are homeomorphs of each other, since both metrics PP' and $F(PP')$ are topologically equivalent.

The class $\Pi(\mathfrak{H})$ contains bounded as well as unbounded functions. Thus $F_2(t)$ and $F_3(t)$, defined above, are bounded. In terms of the function $\gamma(u)$ of (5.2) it is readily seen that $F(t)$ is bounded, if and only if

$$\gamma(0) = \gamma(+0) \qquad \text{and} \qquad \int_{+0}^{1} \frac{d\gamma(u)}{u} \qquad \text{exists.} \tag{5.20}$$

5.6. In [15] it was shown that the function (5.16) $(0 < \kappa < 1)$ enjoys the following further property: If $P_0, \cdots, P_n$ are any $n + 1$ *distinct* points of $F_1(\mathfrak{H})$, the distances $F_1(P_iP_k) = (P_iP_k)^\kappa$ are the edges of a *non-degenerate* simplex in E_n. This property extends as follows to the whole class $\Pi(\mathfrak{H})$. *If $F(t)$ is any element of $\Pi(\mathfrak{H})$, which is not of the form $F(t) = ct$, then any $n + 1$ distinct points of $F(\mathfrak{H})$ are congruent with a non-degenerate n-dim. euclidean simplex.*

Indeed, it suffices to show that if $P_0, \cdots, P_n$ are distinct points of $\mathfrak{H}$, then

$$\sum_0^n \rho_i = 0, \qquad \sum_0^n \rho_i^2 > 0, \qquad \text{imply} \qquad \sum_0^n F^2(P_iP_k)\rho_i\rho_k < 0.$$

Now (5.2) gives

$$F^2(t) = c^2t^2 + \int_{+0}^{\infty} \frac{1 - e^{-t^2u}}{u}\, d\gamma(u), \qquad (c^2 = \gamma(+0) - \gamma(0)), \tag{5.21}$$

and therefore

$$\sum_0^n F^2(P_iP_k)\rho_i\rho_k = c^2 \sum_0^n (P_iP_k)^2\rho_i\rho_k - \int_{+0}^{\infty} \left(\sum_0^n e^{-uP_iP_k^2}\rho_i\rho_k \right) u^{-1}\, d\gamma(u). \tag{5.22}$$

But $\sum (P_iP_k)^2\rho_i\rho_k \leqq 0$ and the integrand in (5.22) is positive by (3.10). There-

fore the vanishing of the quadratic form (5.22) implies that $\gamma(u)$ is constant for $u > 0$, hence $F(t) = ct$.

5.7. It was pointed out in 5.5 that $\Pi(\mathfrak{H})$ contains only one improper transforming function, namely $F(t) \equiv 0$. That the same conclusion may be reached directly, independently of Theorem 6, and for all classes $\Pi(E_m)(2 \leqq m \leqq \infty)$, is shown by the following elementary lemma.

LEMMA 6. *Let $F(t)$ be continuous for $t \geqq 0$, $F(t) \geqq 0$, $F(0) = 0$, $F(t) \not\equiv 0$. If $F(E_m)(2 \leqq m \leqq \infty)$ is imbeddable in a semi-metric space $\mathfrak{R}$, then $F(t) > 0$ for $t > 0$.*

Indeed, let $P \to P'(P \in F(E_m), P' \in \mathfrak{R})$ indicate the mapping of $F(E_m)$ in $\mathfrak{R}$. Denoting by $[P', Q']$ the distance in $\mathfrak{R}$, the isometricity of this imbedding is expressed by the identical relation

$$[P', Q'] = F(PQ), \qquad (P, Q \in E_m). \tag{5.23}$$

Assume that for $t = \tau > 0$ we have $F(\tau) = 0$. Take in E_m an ordinary circle Γ of radius τ, center O, and let A and B be any two points on Γ. Now, by (5.23) we have

$$[O', A'] = F(OA) = F(\tau) = 0, \qquad [O', B'] = F(OB) = F(\tau) = 0,$$

hence ($\mathfrak{R}$ being semi-metric) $O' = A', O' = B'$, and therefore $A' = B'$ or $[A', B'] = 0$. But this implies

$$F(AB) = [A', B'] = 0.$$

As A, B were arbitrary points of Γ, we conclude that $F(t) = 0$ in the interval $0 \leqq t \leqq 2\tau$. Repeating the argument with 2τ, instead of τ, we conclude that $F(t)$ vanishes in the interval $(0, 4\tau)$, and so forth. Hence $F(t) \equiv 0$, in contradiction to our assumption that $F(t) \not\equiv 0$.

Notice that the proof as well as the result fail if $m = 1$. This is already shown by the simplest *closed* euclidean "screw line", namely the circle; for $F(E_1)$, where $F(t) = |\sin t|$ $(0 \leqq t < \infty)$, is imbeddable in E_2. Indeed, taking E_1 to be the real θ-axis with the euclidean metric $|\theta_1 - \theta_2|$, the imbedding of $F(E_1)$ in E_2 is performed by mapping the point $P = \theta$ onto the point P': $x = \frac{1}{2}\cos 2\theta$, $y = \frac{1}{2}\sin 2\theta$, of E_2. For we have identically

$$F^2(P_1P_2) = |\sin|\theta_1 - \theta_2||^2 = \sin^2(\theta_1 - \theta_2) = \tfrac{1}{4}(\cos 2\theta_1 - \cos 2\theta_2)^2 + \tfrac{1}{4}(\sin 2\theta_1 - \sin 2\theta_2)^2 = [P_1', P_2']^2, \text{ hence } [P_1', P_2'] = F(P_1P_2).$$

6. The inner transformations of completely monotone functions

6.1. The simplicial class $\Pi(\mathfrak{H})$ enjoys a further analytical property expressed by the following theorem.

THEOREM 7. *The class of non-negative continuous functions $F(t)$, $(F(0) = 0)$, with the property that if $g(t)$ is an element of $\mathfrak{P}(\mathfrak{H})$, also $g(F(t))$ should be in $\mathfrak{P}(\mathfrak{H})$, coincides with the simplicial class $\Pi(\mathfrak{H})$.*

Indeed, let $F(t) \in \Pi(\mathfrak{H})$ and $g(t) \in \mathfrak{P}(\mathfrak{H})$. In order to prove that $g(F(t)) \in \mathfrak{P}(\mathfrak{H})$, we have to show that if $P_1, \cdots, P_n$ are any points of $\mathfrak{H}$, then

$$\sum_1^n g(F(P_iP_k))\rho_i\rho_k \geqq 0.$$

But this is evident, since there are points Q_i of $\mathfrak{H}$, with $F(P_iP_k) = Q_iQ_k$ (since $F(t) \in \Pi(\mathfrak{H})$), and $\sum g(Q_iQ_k)\rho_i\rho_k \geqq 0$ (since $g(t) \in \mathfrak{P}(\mathfrak{H})$).

To prove the converse, let $F(t)$ be such that $g(t) \in \mathfrak{P}(\mathfrak{H})$ implies $g(F(t)) \in \mathfrak{P}(\mathfrak{H})$. As in particular $g(t) = \exp\{-\lambda t^2\} \in \mathfrak{P}(\mathfrak{H})$, we have

$$e^{-\lambda F^2(t)} \in \mathfrak{P}(\mathfrak{H}), \qquad \text{for all positive } \lambda.$$

By Corollary 2 we now conclude that $F(t) \in \Pi(\mathfrak{H})$, which proves the theorem.

Let us now state Theorem 7 in terms of completely monotone functions by means of Theorem 3. Let $\phi(t)$ be a continuous non-negative function vanishing at the origin. Let us determine all such functions with the property that if $f(t) \in \mathfrak{M}$, also $f(\phi(t))$ should be an element of $\mathfrak{M}$. But if $\phi(t)$ is of this nature and $g(t) \in \mathfrak{P}(\mathfrak{H})$, we may conclude successively that the following relations hold: $g(\sqrt{t}) \in \mathfrak{M}$, $g(\sqrt{\phi(t)}) \in \mathfrak{M}$, $g(\sqrt{\phi(t^2)}) \in \mathfrak{P}(\mathfrak{H})$. Therefore $\sqrt{\phi(t^2)} = F(t) \in \Pi(\mathfrak{H})$, by Theorem 7. In view of (5.14) we thus see that $\phi(t)$ belongs to the class T of §4. This proves the following theorem.

THEOREM 8. *The class of non-negative continuous functions $\phi(t)$, $(\phi(0) = 0)$, with the property that if $f(t)$ is completely monotone for $t \geqq 0$, also $f(\phi(t))$ should be completely monotone for $t \geqq 0$, is identical with the class T of integrals of completely monotone functions:*

$$\phi(t) = \int_{+0}^{t} \psi(t)\, dt, \qquad (t > 0). \tag{6.1}$$

In view of this theorem we may call these functions $\phi(t)$ *inner transformations* of the class $\mathfrak{M}$ of completely monotone functions. In a recent paper ([5], p. 498) S. Bochner has already pointed out this property of the functions (6.1) with the restriction that $\psi(t)$ itself belong to the class $\mathfrak{M}$. This restriction, which amounts to assuming the existence of $\psi(+0)$ rules out such simple elements of T as $\phi(t) = t^\kappa (0 < \kappa < 1)$ for $\psi(t) = \phi'(t) = \kappa/t^{1-\kappa}$ is not bounded near the origin, although $\int_{+0}\psi(t)dt$ obviously exists. However, the principal new contribution of Theorem 8 is not the *enlargement* of Bochner's class of inner transformations, but the statement that the enlarged class T contains *all* such transformations.

6.2. The class T enjoys a further closure property which we state as a corollary.

COROLLARY 3. *If $\phi_1(t)$ and $\phi_2(t)$ are any two elements of T, i.e., of the form* (6.1) *or* (4.4), *also $\phi_1(\phi_2(t))$ belongs to T.*

For if $f(t)$ is completely monotone for $t \geqq 0$, then so is $f(\phi_1(t))$ and therefore also $f(\phi_1(\phi_2(t)))$. Hence $\phi_1(\phi_2(t))$ must belong to T by Theorem 8.

This closure property of T appears here as a consequence of virtually all

previous results of this paper. It seems of sufficient interest to deserve a direct proof to which we now proceed.[22]

We prove first: *If* $\phi(t) \in T$, *then*

$$1 - e^{-\phi(t)} \in T. \tag{6.2}$$

It suffices to show that

$$\psi(t) = \frac{d}{dt}(1 - e^{-\phi(t)}) = e^{-\phi(t)}\phi'(t)$$

is completely monotone for $t > 0$, that is, verifies the inequalities (4.1). Now

$$\psi'(t) = e^{-\phi}(\phi'' - \phi'^2), \qquad \psi''(t) = e^{-\phi}(\phi''' - 3\phi'\phi'' + \phi'^3),$$

and generally

$$\psi^{(n-1)}(t) = \sum (-1)^{0\cdot k_1+1\cdot k_2+\cdots+(n-1)k_n+n-1} \cdot c_{(k)} e^{-\phi}(\phi')^{k_1}(\phi'')^{k_2} \cdots (\phi^{(n)})^{k_n}, \tag{6.3}$$

$$(c_{(k)} > 0),$$

where all terms of this sum have constant weight $k_1 + 2k_2 + \cdots + nk_n = n$. This last statement as well as the fact that the sign before each term has the expression indicated, is readily proved (by induction) by differentiating once both sides of (6.3). Now $\phi'(t)$ being completely monotone for $t > 0$, we have $(-1)^{m-1}\phi^{(m)}(t) \geqq 0$, which, in view of (6.3), clearly implies $(-1)^{n-1}\psi^{(n-1)}(t) \geqq 0$.

We can now complete a proof of Corollary 3. For let

$$\phi_1(t) = \int_0^\infty \frac{1 - e^{-tu}}{u}\, d\gamma(u) \tag{6.4}$$

be a second arbitrary element of T. This integral is the limit, for any fixed $t \geqq 0$, of an appropriate sequence of functions of the form

$$\phi_n^*(t) = \sum_{\nu=1}^{n} \frac{1 - e^{-tu_\nu}}{u_\nu}(\gamma(u_\nu) - \gamma(u_{\nu-1})) \qquad (0 = u_0 < u_1 < \cdots < u_n)$$

where the u_ν depend on n as well. Hence

$$\phi_n^*(\phi(t)) \to \phi_1(\phi(t)) \qquad \text{as} \qquad n \to \infty.$$

As $\phi_n^*(\phi(t)) \in T$, by (6.2), we conclude that also the limit functions $\phi_1(\phi(t))$ is in T, on the basis of the closure property of Lemma 5, §4.

6.3. We conclude this section with two further theorems on the class $\mathfrak{M}$ of functions $f(t)$ which are completely monotone for $t \geqq 0$. We find it convenient to normalize our functions by requiring that $f(0) = 1$. Let $\mathfrak{M}_0$ denote the subclass of elements of $\mathfrak{M}$ thus normalized.

[22] This direct proof was suggested to the author by von Neumann.

THEOREM 9. *The class of elements $f(t)$ of $\mathfrak{M}_0$ with the property that $(f(t))^\lambda$ also belongs to $\mathfrak{M}_0$, for all $\lambda > 0$, is identical with the class of functions of the form*

$$f(t) = e^{-\phi(t)} \tag{6.5}$$

where $\phi(t) \in T$.

THEOREM 10. *If to the previous theorem's assumptions on $f(t)$ we add the further assumption that $f(t)$ be bounded away from zero, $f(t) \geqq \epsilon_0 > 0$, then this new class of functions is identical with the class of functions of the form*

$$f(t) = e^{f_1(t)-f_1(0)}, \tag{6.6}$$

where $f_1(t) \in \mathfrak{M}$.

These theorems are immediate consequences of Corollary 2 and Theorems 3 and 6. Indeed, in order to prove Theorem 9, define $\phi(t)$ by (6.5). Now $(f(t))^\lambda = \exp\{-\lambda\phi(t)\} \in \mathfrak{M}_0$, i.e., $\exp\{-\lambda\phi(t^2)\} \in \mathfrak{P}(\mathfrak{H})$ (Theorem 3), if and only if $\sqrt{\phi(t^2)} \in \Pi(\mathfrak{H})$ (Corollary 2), hence $\phi(t) \in T$ (Theorem 6 and (5.14)). To prove Theorem 10, notice that $f(t)$ is bounded away from zero if and only if $\phi(t)$, of (6.5), is bounded. Now

$$\phi(t) = \int_0^\infty \frac{1 - e^{-tu}}{u}\, d\gamma(u)$$

is bounded, if and only if $\gamma(+0) = \gamma(0)$ and $\int_{+0}^\infty u^{-1}\, d\gamma(u)$ exists. But then

$$\phi(t) = \int_{+0}^\infty \frac{1 - e^{-tu}}{u}\, d\gamma(u) = \int_{+0}^\infty u^{-1}\, d\gamma(u) - \int_{+0}^\infty e^{-tu}u^{-1}\, d\gamma(u) = f_1(0) - f_1(t),$$

where $f_1(t) \in \mathfrak{M}$. Now (6.5) goes over into (6.6).

Incidentally we have proved the following result: *The elements $\phi(t)$ of T which are bounded are of the form*

$$\phi(t) = f(0) - f(t), \qquad (f(t) \in \mathfrak{M}), \tag{6.7}$$

and conversely.

Theorem 9 may also be stated in the following suggestive form: *The elements $f(t)$ of $\mathfrak{M}_0$ which belong to $\mathfrak{M}_0$ together with all their positive powers $(f(t))^\lambda (\lambda > 0)$ are identical with those elements of $\mathfrak{M}_0$ which arise out of the particular function*

$$f(t) = e^{-t} \tag{6.8}$$

by the general inner transformation $t \mid \phi(t)$, where $\phi(t) \in T$.

APPENDIX

7. On metric transforms of euclidean spaces which are isometrically imbeddable in euclidean spaces

7.1. The solution of the problem $\{E_m ; E_n\}$, $(2 \leqq m \leqq n < \infty)$, is given by the following theorem.

THEOREM 11. *If the metric transform $F(E_m)$, of $E_m (m \geqq 2)$ by $F(t)$, is isometrically imbeddable in the euclidean space $E_n (m \leqq n < \infty)$, then necessarily $F(t) = ct$ $(c > 0)$, unless $F(t) \equiv 0$.*

In the proof of this theorem we shall use the solution of the problem $\{E_1 ; E_n\}$ $(n < \infty)$ (von Neumann-Schoenberg, [12]) which is as follows: *$F(E_1)$ is imbeddable in E_n if and only if $F^2(t)$ is of the form*

$$F^2(t) = c^2 t^2 + \sum_{\nu=1}^{r} A_\nu^2 \sin^2 (k_\nu t), \tag{7.1}$$

where $2r + 1 \leqq n$ if $c > 0$, or $2r \leqq n$ if $c = 0$, and c, A_ν, k_ν are non-negative constants.

Let E_1 be the real x-axis with the metric $| x_1 - x_2 |$. The statement that $F(E_1)$ is imbeddable in E_n means that there is a continuous curve $P' = \phi(x)$ $(P' \in E_n , -\infty < x < \infty)$ such that we have identically

$$F^2(| x_1 - x_2 |) = (P_1' P_2')^2 = | \phi(x_1) - \phi(x_2) |^2, \quad (-\infty < x_1, x_2 < \infty), \tag{7.2}$$

in standard vector notation. The curve $P' = \phi(x)$, of E_n, is what was called a *screw line* of E_n, while the corresponding $F(t)$ is a *screw function* of E_n. Thus whenever a continuous function $F(t)$ satisfies an identity of the form $F^2(| t - t' |) = | \phi(t) - \phi(t') |^2$, $(\phi \in E_n)$, for all real t, t', we conclude that $F^2(t)$ is of the form (7.1).

We can now readily prove Theorem 11. Ruling out the trivial case $F(t) \equiv 0$, by Lemma 6, §5.7, we may assume that $F(t) > 0$ for $t > 0$. Consider now a straight line E_1 in E_n. The assumption of Theorem 11 clearly implies that $F(E_1)$ is imbeddable in E_n, hence $F^2(t)$ is of the form (7.1). Now consider in E_n an ordinary circle Γ, of radius $\frac{1}{2}$, referred to the angular parameter θ. Its image in E_n is a simple closed curve $P' = \phi(\theta)$ (of period 2π, $P' \in E_n$). The isometricity of the imbedding of $F(\Gamma)$ in E_n gives

$$F^2(P_1 P_2) = F^2\left(\left| 2 \sin \frac{\theta_1 - \theta_2}{2} \right|\right) = (P_1' P_2')^2 = | \phi(\theta_1) - \phi(\theta_2) |^2,$$

for any real θ_1, θ_2. This shows that also $F\left(\left| 2 \sin \frac{\theta}{2} \right|\right)$ is a screw function of E_n. By (7.1) we therefore have

$$F^2\left(\left| 2 \sin \frac{\theta}{2} \right|\right) = \sum_{\mu=1}^{s} B_\mu^2 \sin^2 (h_\mu \theta), \qquad (2s \leqq n), \tag{7.3}$$

this time without a term in θ^2, since the left side of (7.3) is bounded. From (7.1) and (7.3) we now derive the identity

$$\sum_{\mu=1}^{s} B_\mu^2 \sin^2 (h_\mu \theta) = 4c^2 \sin^2 \frac{\theta}{2} + \sum_{\nu=1}^{r} A_\nu^2 \sin^2 \left(2k_\nu \sin \frac{\theta}{2}\right),$$

from which we readily conclude that for each value of $\nu (\nu = 1, \cdots, r)$ we must have either $A_\nu = 0$, or else $k_\nu = 0$. Now (7.1) reduces to $F^2(t) = c^2 t^2$ and the theorem is established.

7.2. In conclusion we remark that Theorem 11 may be easily proved without reference to the result (7.1) about screw functions of E_n, *if we limit ourselves to the case when* $m = n < \infty$. Omitting details we shall merely sketch the proof. The case $m = n = 1$ is easily disposed of in a manner similar to the treatment of Cauchy's functional relation $F(x + y) = F(x) + F(y)$. Now assume the theorem proved up to the dimension $m - 1$, and let $F(E_m)$ be imbeddable in E_m'. Take two distinct points A and B in E_m and let E_{m-1} be their "perpendicular bisector" in E_m, i.e., the locus of points equidistant from A and B. Now if the imbedding of $F(E_m)$ in E_m' maps A, B into A', B', it is clear that it will map E_{m-1} into the perpendicular bisector E_{m-1}' of A', B'. Therefore $F(t) = ct (c > 0)$.

8. Determination of the rectifiable solutions of the problem $\{E_m ; \mathfrak{H}\}$, $(1 \leqq m < \infty)$

8.1. The class of solutions of the problem $\{E_m ; \mathfrak{H}\}$ may be described in the terminology of §5.2 as the simplicial class $\Pi(E_m)$. As $E_m \subset E_{m+1}$ clearly implies $\Pi(E_m) \supset \Pi(E_{m+1})$, we have before us the following non-increasing sequence of classes

$$\Pi(E_1) \supset \Pi(E_2) \supset \cdots \supset \Pi(E_m) \supset \cdots \supset \Pi(\mathfrak{H}). \tag{8.1}$$

In view of the inequality (5.6) it now becomes apparent that all functions $F(t)$ given by the formula

$$F^2(t) = \int_0^\infty \frac{1 - \Omega_m(tu)}{u^2}\, d\gamma(u), \tag{8.2}$$

where $\gamma(u)$ is non-decreasing for $u \geqq 0$ and such that

$$\int_1^\infty \frac{d\gamma(u)}{u^2} \quad \text{exists,} \tag{8.3}$$

belong to the simplicial class $\Pi(E_m)$. For $\sum_0^n \rho_i = 0$ and (8.2) imply

$$\sum_0^n F^2(P_iP_k)\rho_i\rho_k = -\int_0^\infty \left\{\sum_0^n \Omega_m(P_iP_k \cdot u)\rho_i\rho_k\right\} u^{-2}\, d\gamma(u) \leqq 0, \qquad (P_i \in \bar{\Gamma}_m),$$

since $\Omega_m(tu)$ is positive definite in E_m for $u \geqq 0$.

Conversely, let $F(t)$ be an element of $\Pi(E_m)$, that is, such that the metric transform $F(E_m)$ be imbeddable in $\mathfrak{H}$. A closure property for integrals of the form (8.2), analogous to the closure property of Lemma 5 for the class T, would allow us to conclude by the method of §5.4 that $F^2(t)$ is of the form (8.2). This required closure property has not been established as yet. We shall show very simply, however, that a significant subclass of the class $\Pi(E_m)$ is necessarily given by (8.2) for *bounded* functions $\gamma(u)$. Let us explain first what we mean by a *rectifiable* simplicial function $F(t)$.

We need the following lemma (see Wilson, [21], p. 65, Property II)

LEMMA 7. *If* $F(t) \in \Pi(E_m)$, $F(t) \not\equiv 0$, *then* $F'(0)$ *exists and*

$$0 < F'(0) \leqq \infty. \tag{8.4}$$

We know by Lemma 6, §5.7, that $F(t) > 0$ for $t > 0$, provided $m \geqq 2$. In any case ($m \geqq 1$) it is readily seen that $F(t)$ can not possess arbitrarily small positive zeros, for this would again imply that $F(t) \equiv 0$. This secures a positive quantity τ such that $F(t) > 0$ if $0 < t \leqq \tau$. Now take for E_1 the real x-axis and consider in $F(E_1)$ the simple arc Γ: $0 \leqq x \leqq \tau$. Denote its length in the ordinary sense by s ($s \leqq \infty$), that is, the least upper bound of the length of inscribed polygonal lines. Note that $s \geqq F(\tau) > 0$, hence $s > 0$. Let now δ be a small positive quantity and choose the integer n such that $n\delta \leqq \tau < (n+1)\delta$. The length of the inscribed polygonal line of vertices $x = 0, \delta, 2\delta, \cdots, n\delta, \tau$, is $nF(\delta) + F(\tau - n\delta)$. As $\delta \to 0$, this length converges to the arc-length s; moreover $F(\tau - n\delta) \to 0$, for $\tau - n\delta \to 0$. Hence $nF(\delta) \to s$. From this and the fact that $n\delta \to \tau$, we get

$$\lim_{\delta \to 0} \frac{F(\delta)}{\delta} = \frac{s}{\tau} \qquad (> 0 \text{ and } \leqq \infty), \tag{8.5}$$

which proves our lemma.

The arc Γ of $F(E_1)$ is rectifiable ($s < \infty$) if and only if $F'(0)$ is finite and (8.5) gives $s = \tau F'(0)$. In fact *any* rectifiable arc Γ of E_m of length τ is also rectifiable in $F(E_m)$ and its length in $F(E_m)$ is $s = \tau F'(0)$. For this reason we call $F(t)$ *rectifiable* if $F'(0) < \infty$. Let $\Pi'(E_m)$ denote the subclass of rectifiable elements of $\Pi(E_m)$.

8.2. The rectifiable simplicial functions are determined by the following theorem.

THEOREM 12. *The class $\Pi'(E_m)$ of rectifiable simplicial functions $F(t)$ of $E_m (m \geqq 1)$ is identical with the class of functions such that $F^2(t)$ is of the form* (8.2) *with bounded $\gamma(u)$.*

Indeed, if $\gamma(\infty)$ is finite, then (8.2) and (1.8) give

$$\lim_{t \to 0} \left(\frac{F(t)}{t}\right)^2 = \lim_{t \to 0} \int_0^\infty \frac{1 - \Omega_m(tu)}{t^2 u^2}\, d\gamma(u) = \frac{1}{2m}\, \gamma(\infty) = \sigma^2, \tag{8.6}$$

where we write $2m\sigma^2 = \gamma(\infty)$. Hence $F(t) \in \Pi'(E_m)$.

Conversely, let $F(t) \in \Pi'(E_m)$. By Corollary 1, §5.3, we have $\exp\{-\lambda F^2(t)\} \in \mathfrak{P}(E_m)$ for all $\lambda > 0$, hence by Theorem 1

$$e^{-\lambda F^2(t)} = \int_0^\infty \Omega_m(tu)\, d\alpha(u, \lambda), \qquad (t, \lambda > 0),$$

where $\alpha(u, \lambda)$ is a family of monotone functions with $\alpha(0, \lambda) = 0$, $\alpha(\infty, \lambda) = 1$. As in §5.4 we may now write

$$\frac{1 - e^{-\lambda F^2(t)}}{\lambda} = \int_0^\infty \frac{1 - \Omega_m(tu)}{\lambda}\, d\alpha(u, \lambda) = \int_0^\infty \frac{1 - \Omega_m(tu)}{u^2}\, d\beta(u, \lambda), \tag{8.7}$$

where

$$\beta(u, \lambda) = \frac{1}{\lambda} \int_0^u u^2\, d\alpha(u, \lambda). \tag{8.8}$$

On the other hand the finiteness of $\sigma = F'(0)$ implies $F(t) < (\sigma + \epsilon)t$, for t sufficiently small, and therefore

$$t^2(\sigma + \epsilon)^2 > F^2(t) \geqq \frac{1 - e^{-\lambda F^2(t)}}{\lambda}.$$

By (8.7) we now have

$$(\sigma + \epsilon)^2 > \int_0^\infty \frac{1 - \Omega_m(tu)}{u^2 t^2}\, d\beta(u, \lambda),$$

for t sufficiently small. As $t \to 0$ we now get $\beta(\infty, \lambda) \leqq 2m(\sigma + \epsilon)^2$, showing that the family of monotone functions (8.8) is bounded uniformly for all $\lambda > 0$. For a suitable sequence $\lambda_n \to 0$ we shall therefore have $\beta(u, \lambda_n) \to \gamma(u)$ in all continuity points of $\gamma(u)$. Now, if t is kept fast in (8.7) while $\lambda = \lambda_n \to 0$, we get in the limit the desired representation (8.2).[23]

Let $F(t)(\not\equiv 0)$ be a rectifiable element of $\Pi(E_m)$ and let us assume $m \geqq 2$. The space $F(E_m)$ is a homeomorph of E_m and its geodesics are the straight lines of E_m. The geodesic distance, or intrinsic distance, $l(P, Q)$ between two points of $F(E_m)$ (in the terminology and notation of Wilson, [20], p. 420) is $l(P, Q) = F'(0) \cdot PQ$. Thus by passing in $F(E_m)$ from its metric $F(PQ)$ to the intrinsic metric $l(P, Q)$ which it generates, we merely get a similar image of the original space E_m, the ratio of similitude being $F'(0)$. Notice furthermore that an arc of a geodesic line in $F(E_m)$ (i.e., a straight segment of the underlying E_m) never loses its minimizing property of an absolute minimum no matter how far it is extended in both directions. In the isometric image Σ of $F(E_m)$ in $\mathfrak{H}$, the geodesics of $F(E_m)$ go over in screw lines of $\mathfrak{H}$, all of which are congruent among themselves. In fact any two congruent figures in E_m are mapped in congruent figures of Σ. To any rigid motion of E_m in itself corresponds a rigid motion of Σ in itself.

8.3. We close the paper with the following analogue of Theorem 6′.

THEOREM 12′. *A necessary and sufficient condition that the non-negative continuous function $F(t)$, vanishing at the origin, be an element of $\Pi'(E_m)$, is that $F^2(\sqrt{t})$ be continuously differentiable for $t \geqq 0$ and, on setting*

$$\psi(t) = \frac{d}{dt} F^2(\sqrt{t}), \tag{8.9}$$

the function $\psi(t^2)$ be positive definite in E_{m+2}.

Indeed, denoting by c_m, c_m', c_m'' positive constants and setting $k = (m - 2)/2$, we have by (1.8) (see [18], p. 45)

$$\Omega_m'(r) = c_m(d/dt)(J_k(r)r^{-k}) = -c_m J_{k+1}(r)r^{-k},$$

hence

$$-\Omega_m'(r)r^{-1} = c_m' \Omega_{m+2}(r).$$

[23] The basic idea of this proof was suggested by the proof of Theorem 27, p. 91, in H. Cramér's monograph [6]. References are there found to the work of Kolmogoroff, De Finetti, and P. Lévy.

An element $F(t)$ of $\Pi'(E_m)$ is given by (8.2) with a bounded $\gamma(u)$. Hence

$$F^2(\sqrt{t}) = \int_0^\infty \frac{1 - \Omega_m(\sqrt{(tu)})}{u}\, d\gamma(\sqrt{u}).$$

By differentiation we get

$$\psi(t) = \frac{d}{dt} F^2(\sqrt{t}) = -\int_0^\infty \frac{\Omega_m'(\sqrt{(tu)})}{2\sqrt{(tu)}}\, d\gamma(\sqrt{u}) = c_m'' \int_0^\infty \Omega_{m+2}(\sqrt{(tu)})\, d\gamma(\sqrt{u}),$$

or

$$\psi(t^2) = c_m'' \int_0^\infty \Omega_{m+2}(tu)\, d\gamma(u).$$

Hence $\psi(t^2) \in \mathfrak{P}(E_{m+2})$, by Theorem 1. The converse is proved by retracing our steps.

We discuss only one particular example, namely

$$F_5(t) = (1 - \Omega_3(t))^{\frac{1}{2}} = (1 - (\sin t/t))^{\frac{1}{2}}. \tag{8.10}$$

By (8.2) we have $F_5(t) \in \Pi'(E_3)$. As above we get

$$\psi(t) = \frac{d}{dt} F_5^2(\sqrt{t}) = c_3'' \Omega_5(\sqrt{t})$$

which is *not* completely monotone for $t > 0$, since $\Omega_5(r)$ has infinitely many positive zeros. Hence $F_5(t)$ is not an element of $\Pi(\mathfrak{H})$, by Theorem 6′. Thus $F_5(E_3)$ is imbeddable in $\mathfrak{H}$, but $F_5(\mathfrak{H})$ is not imbeddable in $\mathfrak{H}$. A further argument, which we omit, would show that already $F_5(E_4)$ is not congruent to a subset of $\mathfrak{H}$.

COLBY COLLEGE,
WATERVILLE, MAINE.

REFERENCES

[1] S. BERNSTEIN, *Sur les fonctions absolument monotones*, Acta Mathematica, vol. 52 (1929), pp. 1–66.

[2] L. M. BLUMENTHAL, *Remarks concerning the euclidean four-point property*, Ergebnisse eines math. Koll., issue 7, Vienna 1936, pp. 8–10.

[3] ——— *Distance Geometries*, University of Missouri Studies, vol. 13 (1938), 142 pp.

[4] S. BOCHNER, *Monotone Funktionen, Stieltjes Integrale und harmonische Analyse*, Math. Annalen, vol. 108 (1933), pp. 378–410.

[5] ——— *Completely monotone functions of the Laplace operator for torus and sphere*, Duke Math. Journal, vol. 3 (1937), pp. 488–502.

[6] H. CRAMÉR, *Random variables and probability distributions*, Cambridge Tract no. 36, Cambridge, 1937.

[7] F. HAUSDORFF, *Summationsmethoden und Momentfolgen* II, Math. Zeitschrift, vol. 9 (1921), pp. 280–299.

[8] E. HELLY, *Über lineare Funktionaloperationen*, Sitzungsber. der Wiener Akad., vol. 121, IIa (1912), pp. 265–297.

[9] K. MENGER, *Die Metrik des Hilbertschen Raumes*, Anzeiger der Akad. der Wiss. in Wien, Math.-Nat. Klasse, vol. 65 (1928), pp. 159–160.

[10] K. MENGER, *New foundations of euclidean geometry*, Amer. J. of Math., vol. 53 (1931), pp. 721–745.

[11] E. H. MOORE AND R. W. BARNARD, *General Analysis*, Part I, Memoirs of the American Phil. Soc., vol. 1, Philadelphia, 1935.

[12] JOHN VON NEUMANN AND I. J. SCHOENBERG, *Fourier integrals and metric geometry*, to appear in the Transactions of the Amer. Math. Soc.

[13] I. J. SCHOENBERG, *On finite-rowed systems of linear inequalities in infinitely many variables*, Trans. of the Amer. Math. Soc., vol. 34 (1933), pp. 594–619.

[14] ——— *Remarks to Maurice Fréchet's article . . .*, these Annals, vol. 36 (1935), pp. 724–732.

[15] ——— *On certain metric spaces arising from euclidean spaces by a change of metric and their imbedding in Hilbert space*, these Annals, vol. 38 (1937), pp. 787–793.

[16] ——— *Metric spaces and positive definite functions*, to appear in the Trans. of the Amer. Math. Soc.

[17] E. C. TITCHMARSH, *The theory of functions*, Oxford, 1932.

[18] G. N. WATSON, *Theory of Bessel functions*, Cambridge, 1922.

[19] D. V. WIDDER, *Necessary and sufficient conditions for the representation of a function as a Laplace integral*, Trans. of the Amer. Math. Soc., vol. 33 (1932), pp. 851–892.

[20] W. A. WILSON, *On rectifiability in metric spaces*, Bulletin of the Amer. Math. Soc., vol. 38 (1932), pp. 419–426.

[21] ——— *On certain types of continuous transformations of metric spaces*, Amer. J. of Math., vol. 57 (1935), pp. 62–68.

A REMARKABLE PROPERTY OF THE RIEMANN-CHRISTOFFEL TENSOR IN FOUR DIMENSIONS

By Cornelius Lanczos

(Received February 10, 1938)

Introduction. If the geometry of nature is Riemannian and the field equations of this geometry are controlled by a scale-invariant action principle, there are four a priori possible and algebraically independent invariants which may enter in the integrand of the action principle. This abundance of invariants hampers the mathematical development and the logical appeal of the theory. The present paper shows that two of these invariants are inactive in the formation of field equations and thus may be omitted. Only the two invariants

$$I_1 = R_{\alpha\beta}R^{\alpha\beta} \quad \text{and} \quad I_2 = R^2$$

remain effective which are formed with the help of the contracted curvature tensor R_{ik} alone, while the complete curvature tensor $R_{\alpha\beta\mu\nu}$ drops out from the action principle.

1. In the attempt to find the fundamental geometrical laws of nature, the general possibilities can be greatly reduced by a small number of reasonable a priori assumptions:

a) The geometry of nature shall be of the *Riemannian type.*

b) This geometry shall be characterized by *field equations* for the metrical tensor.

c) In view of the universal appearance of the principle of least action in all branches of physics: the field equations shall be deducible from an *action principle.*

d) The existence of an action principle reduces the problem of geometry to the determination of *one fundamental invariant* I, viz. the integrand of the action principle:

$$\delta \int I d\tau = 0, \tag{1.1}$$

($d\tau$ = four-dimensional volume element) that controls the field equations of geometry. This fundamental invariant has to be composed out of the fundamental quantities of Riemannian geometry and these are, in addition to the g_{ik}, the components of the curvature tensor $R_{\alpha\beta\mu\nu}$ called the Riemann-Christoffel tensor. It is feasible to assume that the fundamental invariant I shall be merely an *algebraic* function of the components of the curvature tensor, not involving any differentiation. As far as the *form* of this function is concerned,

the principle of "scale-independence" may be assumed: the value of the action-integral that has to be minimized should not depend on the arbitrary units employed in measuring the lengths of the space-time manifold. Owing to this principle, the invariant I has to be a *quadratic* function of the curvature components $R_{\alpha\beta\mu\nu}$.[1]

2. The conditions a), b), c), and d) permit one to restrict to a large degree, in a purely logical fashion, the laws of geometry presumably realized in the physical universe. In a number of papers[2] the author investigated the general nature of field equations which are derived from an action principle with a quadratic invariant. He has also applied the Hamiltonian method of doubling the number of independent variables of the given action principle, treating the g_{ik} and the R_{ik} as two independent sets of variables and considering the field equations, in generalization of Einstein's gravitational equations, as a mutual interaction of matter tensor and metrical tensor.[3] There are indications in this theory for the appearance of the vector potential and thus for the possibility of explaining the electromagnetic phencmena. But the mathematical difficulty of obtaining "regular solutions" of the field equations, to be correlated to the physical behavior of single particles, is still a serious objection against the plausibility of these considerations.[4]

There is, however, an even more serious obstacle that has handicapped the

---

[1] As it is well known, H. Weyl has developed an ingenious theory (see Ann d. Physik *59*, 101, (1919); W. Pauli: *Relativitäts-theorie* (Teubner, 1921), p. 759), which generalizes the Riemannian basis of geometry and builds up geometry in a "purely infinitesimal" fashion. In this geometry the ordinary principle of invariance is completed by the principle of "gauge-invariance," which requires invariance with respect to the substitution:

$$\bar{g}_{ik} = \lambda g_{ik} \tag{1.2}$$

where λ is an arbitrary function of the coördinates. If we do not forsake the basis of Riemannian geometry, which postulates the existence of an infinitesimal *and transportable* calibrated yard-stick, the principle of gauge-invariance has still its logical significance in a *limited* sense, viz. for *constant* λ. The substitution (1.2) with constant λ has the significance of *changing the scale of calibration* of the infinitesimal yard-stick. Since this scale may be chosen arbitrarily, the factor λ is undetermined. In general, an arbitrary invariant of Riemannian geometry will be a "dimensioned" quantity, i.e. it will depend on the value of the scale-constant λ. It is feasible to assume that the fundamental integral, which has to be minimized according to the action principle, shall have the "dimension zero," i.e. it shall not depend on the arbitrary scale-factor we employ in measuring lengths of the space-time manifold. Otherwise any arbitrary small or large value can be assigned to the fundamental integral by merely changing the unit of the length—except if that value happens to be zero.

[2] Phys. Rev. *39*, 716 (1932); Zeits. f. Physik *73*, 147 (1931) and *75*, 63 (1932).

[3] Zeits. f. Physik. *96*, 76 (1935).

[4] Some recent advances in the field of approximation methods led to a new approach of the problem and seem to indicate that the required "proper solutions" are actually present. The author hopes to report about these results in the near future.

progress along these lines considerably. The investigations of the author are based on a linear combination of the two invariants

$$I_1 = R_{\alpha\beta}R^{\alpha\beta}, \tag{2.1}$$

and

$$I_2 = R^2. \tag{2.2}$$

These are, however, not the *only* invariants satisfying the condition of gauge-invariance. Also the following invariant satisfies all conditions:

$$I_3 = R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}. \tag{2.3}$$

This invariant involves the complete curvature tensor of 4^{th} order and seems mathematically of a much more cumbersome nature than the invariants (2.1) and (2.2). This, however, is no inherent reason for excluding it from the participation in the action integral.[5]

Even more discouraging is the fact that still more invariants of quadratic nature can be constructed if we make use of the completely anti-symmetric "determinant tensor"

$$\delta^{\alpha\beta\mu\nu} = \frac{1}{\sqrt{g}}\epsilon^{\alpha\beta\mu\nu}, \tag{2.4}$$

where the symbol $\epsilon^{\alpha\beta\mu\nu}$ has the value $+1$ for any even permutation of the four indices α, β, μ, ν and the value -1 for any odd permutation, provided that no two indices are equal. In the latter case $\epsilon^{\alpha\beta\mu\nu}$ vanishes.

With the help of the tensor (2.4) we can form the "simply dual" and the "doubly dual" curvature tensors:

$$\overset{*}{R}_{ik}{}^{mn} = R_{ik\alpha\beta}\delta^{\alpha\beta mn}, \tag{2.5}$$

and

$$\overset{**}{R}{}^{ikmn} = R_{\alpha\beta\mu\nu}\delta^{\alpha\beta ik}\delta^{\mu\nu mn}, \tag{2.6}$$

which give rise to two more invariants:

$$K_1 = R_{\alpha\beta\mu\nu}\overset{*}{R}{}^{\alpha\beta\mu\nu}, \tag{2.7}$$

and

$$K_2 = R_{\alpha\beta\mu\nu}\overset{**}{R}{}^{\alpha\beta\mu\nu}. \tag{2.8}$$

---

[5] The alternative combination $R_{\alpha\beta\mu\nu}R^{\alpha\mu\beta\nu}$, mentioned by Weyl (see reference 1, p. 133), is reducible to (2.3) owing to the algebraic identity:

$$R_{\alpha\beta\mu\nu} + R_{\alpha\mu\nu\beta} + R_{\alpha\nu\beta\mu} = 0 \tag{2.9}$$

One can prove that similarly also in case of the two invariants (2.7) and (2.8), in consequence of the same identity, the *transposition of indices does not produce new invariants.*

Hence there are altogether 5 apparently independent invariants which satisfy all our conditions and we may choose an arbitrary linear combination of them, with some numerical factors, as our action integrand I. This abundance of equivalent elements, among which no a priori choice seems to be possible, impairs greatly the logical persuasiveness of the theory.

The present investigation shows that this surplus of invariants is only apparent. We shall see that *the invariants K_1 and K_2 do not contribute any terms to the field equations* since their variation reduces to a mere boundary term. Moreover, we shall prove *that the invariant I_3, as far as field equations are concerned, is reducible to the two invariants I_1 and I_2*. These invariants were chosen previously owing to their property to depend only on the contracted curvature tensor R_{ik} of Einstein and not on the general Riemannian tensor $R_{\alpha\beta\mu\nu}$ of 4^{th} order. The present study yields the deeper evidence for the fundamental nature of Einstein's curvature tensor R_{ik}, proving that those invariants which involve the complete curvature tensor of the 4^{th} order are reducible, as far as the forming of field equations is concerned, to the two simpler invariants I_1 and I_2 which are constructed with the help of the contracted tensor R_{ik} alone.

3. The following formulae belong to the common arsenal of tensor-calculus and may be found more or less completely in almost any of the common textbooks on tensor-calculus or relativity. They are listed here for later references. The notations are self-explanatory and agree with the standard methods of writing tensor-equations. In lack of a universally accepted practical symbol for the process of "covariant differentiation" the ordinary symbol of partial differentiation: $\partial/\partial x_k$ shall be adopted for this purpose, in view of the fact that the ordinary differentiation will not enter in our present considerations. In accordance with the "sum-convention" of Einstein, equal indices shall automatically mean summation indices.

a) The Gaussian integral transformation:

$$\int \frac{\partial V^\alpha}{\partial x_\alpha}\, dt = \text{surface integral.} \tag{3.1}$$

b)
$$\int \frac{\partial A_{\alpha\beta\ldots\mu}}{\partial x_\nu} B^{\alpha\beta\ldots\mu\nu}\, d\tau = \text{surface integral} - \int A_{\alpha\beta\ldots\mu} \frac{\partial B^{\alpha\beta\ldots\mu\nu}}{\partial x_\nu}\, d\tau. \tag{3.2}$$

c) Bianchi's identity for the Riemann-Christoffel tensor:

$$\frac{\partial R_{\alpha\beta\mu\nu}}{\partial x_\rho} + \frac{\partial R_{\alpha\beta\nu\rho}}{\partial x_\mu} + \frac{\partial R_{\alpha\beta\rho\mu}}{\partial x_\nu} = 0. \tag{3.3}$$

d) As it is well known, the quantities Γ^α_{ik} do not form the components of a tensor of third order. But the *variation* of the Γ^α_{ik}, produced by an infinitesimal variation of the g_{ik}, is actually a genuine tensor of third rank, covariant and symmetric in i, k, contravariant in α:

$$\delta\Gamma^\alpha_{ik} = \gamma^\alpha_{ik}. \tag{3.4}$$

e) The variation of the curvature tensor $R^{\alpha}_{\beta\mu\nu}$ can be expressed by means of the tensor γ^{α}_{ik} in the following form:

$$\delta R^{\alpha}_{\beta\mu\nu} = \frac{\partial \gamma^{\alpha}_{\beta\mu}}{\partial x_\nu} - \frac{\partial \gamma^{\alpha}_{\beta\nu}}{\partial x_\mu}. \tag{3.5}$$

f) The "dual transposition" of an anti-symmetric pair of indices may be obtained from the following table:

$$\begin{matrix} 12 & 34 \\ 13 & 42 \\ 14 & 23. \end{matrix} \tag{3.6}$$

The table reads in both directions and contains thus all possible combinations of two different indices. Exchange of the sequence of an index-pair means the corresponding exchange in the correlated pair.

The process of "dual transposition" of an anti-symmetric pair of indices shall be denoted by *underlining* these indices. Thus e.g. $R_{\alpha\beta\underline{12}}$ shall mean $R_{\alpha\beta 34}$, etc. With this notation we may write:

$$\overset{*}{R}_{\alpha\beta}{}^{\mu\nu} = \frac{1}{\sqrt{g}} R_{\alpha\beta\underline{\mu\nu}} \tag{3.7}$$

$$\overset{**}{R}{}^{\alpha\beta\mu\nu} = \frac{1}{g} R_{\underline{\alpha\beta}\underline{\mu\nu}} \tag{3.8}$$

4. We deal at first with the two invariants K_1 and K_2 and show that the terms, obtained from them by variation of the g_{ik}, all vanish identically. This is in the case of K_1 due to Bianchi's differential identity, while in the case of K_2 we have to add the algebraic symmetry-properties of the Riemann-Christoffel tensor, together with the fact that the number of variables is 4.

The Bianchi-identity (3.3) may be written for the special case of 4 dimensions in the following form:

$$\frac{\partial R_{ik\alpha\beta}}{\partial x_\gamma} \delta^{\alpha\beta\gamma m} = 0. \tag{4.1}$$

Due to this relation we get for the two dual tensors (2.5) and (2.6):

$$\frac{\partial \overset{*}{R}_{ikm}{}^{\alpha}}{\partial x_\alpha} = 0, \tag{4.2}$$

$$\frac{\partial \overset{**}{R}_{ikm}{}^{\alpha}}{\partial x_\alpha} = 0. \tag{4.3}$$

We now consider the action integral

$$A = \int R_{\alpha\beta\mu\nu} \overset{*}{R}{}^{\alpha\beta\mu\nu} \, d\tau. \tag{4.4}$$

Since the volume-element $d\tau$ has the significance

$$d\tau = \sqrt{g}\, dx_1 \cdots dx_4, \tag{4.5}$$

we may write (4.4), making use of (3.7), in the following form:

$$A = \int R^{\alpha}{}_{\beta\mu\nu} R_{\alpha}{}^{\beta}{}_{\mu\nu}\, dx_1 \cdots dx_4. \tag{4.6}$$

Owing to the complete mutual symmetry of the two factors the variation of the second factor reproduces the result of the variation of the first factor. Thus it suffices to vary the first factor alone and multiply the result by 2. Applying the formula (3.5) for the variation of the curvature tensor and making use of the integral-transformation (3.2) we obtain:

$$\begin{aligned} \delta A &= 2 \int \left(\frac{\partial \gamma^{\alpha}_{\beta\mu}}{\partial x_\nu} - \frac{\partial \gamma^{\alpha}_{\beta\nu}}{\partial x_\mu} \right) R_{\alpha}{}^{\beta}{}_{\mu\nu}\, dx_1 \cdots dx_4 \\ &= 4 \int \frac{\partial \gamma^{\alpha}_{\beta\mu}}{\partial x_\nu} R_{\alpha}{}^{\beta}{}_{\mu\nu}\, dx_1 \cdots dx_4 \\ &= \text{boundary term} - 4 \int \gamma^{\alpha}_{\beta\mu} \frac{\partial \overset{*}{R}_{\alpha}{}^{\beta\mu\nu}}{\partial x_\nu}\, d\tau. \end{aligned} \tag{4.7}$$

The second term vanishes owing to (4.2). Thus δA reduces to a boundary term and does not yield any terms toward the field equations.

We now turn to the second invariant K_2 and handle the variation problem in a similar manner. The action integral

$$A = \int R_{\alpha\beta\mu\nu} \overset{**}{R}{}^{\alpha\beta\mu\nu}\, d\tau \tag{4.8}$$

may now be written as follows:

$$A = \int R_{\alpha\beta\mu\nu} R_{\underline{\alpha}\underline{\beta}\mu\nu}\, dx_1 \cdots dx_4 = R^{\rho}{}_{\beta\mu\nu} R^{\sigma}{}_{\underline{\beta}\mu\nu} \frac{g_{\rho\alpha} g_{\sigma\underline{\alpha}}}{\sqrt{g}}\, dx_1 \cdots dx_4. \tag{4.9}$$

Again the variation of the first and the second factor yields the same, due to the symmetry of their mutual relation. The process (4, 7) again yields a pure boundary term in view of the fact that not only $\overset{*}{R}_{\alpha\beta\mu\nu}$ but also $\overset{**}{R}_{\alpha\beta\mu\nu}$ satisfies the divergence equation (4.3). Hence only the variation of the *third* factor will contribute something to the field equations. Here no derivatives of the g_{ik} enter any more, and thus the field equations deduced from the invariant K_2 have the remarkable property of being not higher than *second* order in the g_{ik}. We will prove, however, that *these terms all vanish identically*, owing to the algebraic symmetry-properties of the tensor $R_{\alpha\beta\mu\nu}$.

Performing the variation of the g_{ik} in the third factor, we obtain the following symmetric tensor of second order:

$$S_{ik} = R_{i\alpha\beta\gamma} \overset{**}{R}_{k}{}^{\alpha\beta\gamma} + R_{k\alpha\beta\gamma} \overset{**}{R}_{i}{}^{\alpha\beta\gamma} - \tfrac{1}{2} K_2 g_{ik}. \tag{4.10}$$

Now a tensor-equation may be proved in *any* reference-system. Thus we may introduce a local reference-system in which the g_{ik} have the normal values δ_{ik}. Moreover, by a suitable rotation of our reference-system, the symmetric tensor S_{ik} may be transformed into a *purely diagonal* form. Hence it suffices to show that all the *diagonal* terms of the expression (4.10) vanish. This guarantees the vanishing of the *complete* tensor.

Now in our local reference system co-variant and contra-variant components are equal and the diagonal terms of S_{ik} may be written as follows:

$$\tfrac{1}{2}S_{ii} = R_{i\alpha\beta\gamma}\overset{**}{R}_{i\alpha\beta\gamma} - \tfrac{1}{4}K_2 = R_{i\alpha\beta\gamma}R_{i\underline{\alpha}\underline{\beta}\underline{\gamma}} - \tfrac{1}{4}K_2 \tag{4.11}$$

The index i is here exceptionally *no* sum-index, but a fixed index which assumes successively the values 1, 2, 3, 4, while α, β, γ are, as usual, sum-indices.

We examine particularly the expression

$$\rho_i = R_{i\alpha\beta\gamma}R_{i\underline{\alpha}\underline{\beta}\underline{\gamma}} ,$$

the second term of (4.11) being the same for all the four S_{ii}. Let us evaluate the contributions of the different curvature-components of ρ_i *separately*. We may have an index-combination of the type R_{1212} with two equal pairs of indices. This component combines with R_{3434} so that the effect of these two components has to be considered simultaneously. Now in the first row ($i = 1$) the product $(12, 12)\cdot(34, 34)$ enters[6], omitting for a moment the letter R and denoting the various components of R merely by the corresponding index-combinations. In the second row ($i = 2$) the same product enters, due to the combination $(21, 21)\cdot(43, 43)$. The same happens in the third and fourth rows, with changed sequence of the factors. Thus all the four ρ_i are equal.

We prove the same for a combination of the type (12, 13); (only *two* equal indices). Here the dual partner is (34, 42). In the first row we get two terms since 1 may combine with either 2 or 3:

$$(12, 13)\cdot(34, 42) + (13, 12)\cdot(42, 34) = 2\cdot(12, 13)\cdot(34, 42).$$

In the second row 2 may combine with either 1 or 4 and we obtain:

$$(21, 13)\cdot(43, 42) + (24, 34)\cdot(12, 31) = 2\cdot(12, 13)\cdot(34, 42).$$

The result for the third and fourth row is similar. Again all the four ρ_i are the same.

Finally we consider a combination of the type (12, 34), where all the four indices are different. Here the square of (12, 34) enters into the first row, but the same happens in all the other rows. The contributions to all four rows are the same.

Hence we have proved for any possible index-combinations that all the four ρ_i

[6] Actually also (12, 21)·(34, 43) should be considered which only duplicates the result. We consistently omit here and in the next similar instances the interchange of the second pair of indices.

come out as equal and then the same holds, due to (4.11), for the four S_{ii}. We may put

$$S_{ii} = C. \tag{4.13}$$

But the expression (4.10) shows directly that the scalar

$$S_\alpha^\alpha = 0. \tag{4.14}$$

This means for our reference system:

$$4C = 0. \tag{4.15}$$

We thus arrive at the conclusion that all the diagonal components of the tensor S_{ik} vanish which leads to the complete vanishing of the tensor S_{ik}.

The result of this paragraph is that *neither the invariant K_1 nor the invariant K_2 is able to contribute any terms to the field-equations.*

5. The tensor $\overset{**}{R}_{\alpha\beta\mu\nu}$ is not independent of the original tensor $R_{\alpha\beta\mu\nu}$ but can be reduced to it without referring to the anti-symmetric tensor $\delta^{\alpha\beta\mu\nu}$. Let us form the difference

$$R_{\alpha\beta\mu\nu} - \overset{**}{R}_{\alpha\beta\mu\nu}. \tag{5.1}$$

This tensor has only 9 independent components and can be produced explicitly with the help of the contracted tensor R_{ik}. We can prove by direct inspection in a local reference-system $g_{ik} = \delta_{ik}$ the validity of the following tensor relation:

$$R_{\alpha\beta\mu\nu} - \overset{**}{R}_{\alpha\beta\mu\nu} = B_{\alpha\mu}g_{\beta\nu} - B_{\alpha\nu}g_{\beta\mu} + B_{\beta\nu}g_{\alpha\mu} - B_{\beta\mu}g_{\alpha\nu}, \tag{5.2}$$

where we have put

$$B_{ik} = R_{ik} - \tfrac{1}{4}Rg_{ik}. \tag{5.3}$$

Hence

$$\overset{**}{R}_{\alpha\beta\mu\nu} = R_{\alpha\beta\mu\nu} - B_{\alpha\mu}g_{\beta\mu} + B_{\alpha\nu}g_{\beta\mu} - B_{\beta\nu}g_{\alpha\mu} + B_{\beta\mu}g_{\alpha\nu}. \tag{5.4}$$

Multiplying by $R^{\alpha\beta\mu\nu}$ we get the following linear relation between the invariants I_1, I_2, I_3 and K_2:

$$K_2 = I_3 - 4R^{\alpha\mu}B_{\alpha\mu} = I_3 - 4I_1 + I_2. \tag{5.5}$$

The fact that the field equations deduced from K_2 vanish identically may now be expressed in the following manner: *The field equations deduced from the action integral*

$$\int I_3 d\tau \tag{5.6}$$

are identical with the field-equations deduced from the action-integral

$$\int (4I_1 - I_2)d\tau. \tag{5.7}$$

6. Conclusion. If we want to investigate the complete group of Riemannian geometries deducible from an action principle with an integrand which is quadratic in the curvature components and thus scale-invariant, we have to start out with a general expression of the following form:

$$\delta \int (I_1 + \alpha I_2 + \beta I_3 + \gamma K_1 + \epsilon K_2) d\tau = 0 \tag{6.1}$$

with the five invariants I_1, I_2, I_3, K_1, K_2 and four numerical constants α, β, γ, ϵ. Due to a linear relation between these five invariants I_3 may be eliminated and thus the action integral reduced to but four independent invariants. It has been shown that for any variations between definite limits we get identically:

$$\delta \int K_1 d\tau = 0, \tag{6.2}$$

$$\delta \int K_2 d\tau = 0, \tag{6.3}$$

so that the action principle (6.1) reduces in fact to the simple form

$$\delta \int (I_1 + \alpha I_2) d\tau = 0. \tag{6.4}$$

The two remaining invariants

$$I_1 = R_{\alpha\beta} R^{\alpha\beta}, \qquad I_2 = R^2, \tag{6.5}$$

are characterized by the property that they are formed exclusively by means of the contracted curvature tensor R_{ik} while the general Riemann-Christoffel tensor $R_{\alpha\beta\mu\nu}$ does not enter any more into the action-principle. Thus the difficulty of an abundance of invariants, coupled with each other by arbitrary numerical constants, is removed and the action integral reduced to a form in which but *one* unknown numerical constant remains.

This result confirms in a very satisfactory manner the fundamental significance of the contracted curvature tensor of Einstein for the geometry of nature. It also confirms the logical reasonableness of the attempt to deduce the field equations of nature from a scale-invariant action principle.

PURDUE UNIVERSITY,
LAFAYETTE, IND.

ANNALS OF MATHEMATICS
Vol. 39, No. 4, October, 1938

MONOTONE TRANSFORMATIONS OF TWO-DIMENSIONAL MANIFOLDS

BY J. H. ROBERTS AND N. E. STEENROD

(Received June 2, 1938)

INTRODUCTION

If M and G are topological spaces, a continuous transformation $T(M) = G$ of M onto G is said to be *monotone* if the inverse image of each point of G is connected. In this paper we consider the case M = a compact 2-dimensional manifold without boundary, and we give a characterization of the class of all images G of M under monotone transformations. This problem has already been solved by R. L. Moore[1] in the case M = a 2-sphere. His result is that G is a cactoid,[2] and every cactoid is a monotone image of a 2-sphere. Our results are highly analogous (see Theorem 5).

If $T(M) = G$ is monotone, the set of inverse images of points of G forms an upper semi-continuous collection of continua filling M. Conversely if G is an upper semi-continuous collection of continua filling M, the set G can be so topologized that the transformation sending $P \in M$ into the continuum of G containing P is a continuous monotone transformation of M onto G. Thus the study of monotone transformations is equivalent to the study of upper semi-continuous collections of continua.

In addition to the characterization of G in the general case we find the effect on G of certain restrictions on the elements of G, considered as continua on M. Theorems 1, 2, 3, respectively give conditions both necessary and sufficient that G be (1) homeomorphic to M, (2) a 2-manifold, and (3) a generalized cactoid.

I. CONTINUA ON A TWO-MANIFOLD

LEMMA 1. *Let M be a compact 2-manifold without boundary, g a continuum on M and D a component of $M - g$. Then D contains a finite closed connected complex K whose* mod 2 *boundary is the sum of a finite number $\Gamma_1, \cdots, \Gamma_s$ of pairwise disjoint simple closed curves, and $D - K = H_1 + \cdots + H_s$, where the H_i are mutually exclusive open cylinders and the boundary of H_i is Γ_i plus a subset of g.*

Let D be subdivided so as to be an infinite complex $\bar{K}$. The mod 2 1-dimensional Betti number of $\bar{K}$ based on finite chains is finite. To see this, let Γ be any 1-cycle of $\bar{K}$ bounding a chain in M. Then Γ separates M into two parts

[1] R. L. Moore, *Concerning upper semi-continuous collections*, Monatshefte für Math. und Physik, vol. 36 (1929), pp. 81–88.

[2] A *cactoid* is a continuous curve whose every maximal cyclic element is a 2-sphere.

M_1, M_2. As g is connected it lies wholly in one, say M_1. This means that M_2 is in D. Hence Γ bounds in $\bar{K}$. Thus the Betti number of $\bar{K}$ is less than or equal to that of M.

We may therefore choose a finite subcomplex K_0 of $\bar{K}$ containing a basis for the 1-cycles of $\bar{K}$. By adding additional cells to K_0 if necessary, we can readily obtain a connected K_0 and such that each 1-cell of K_0 is on the boundary of at least one 2-cell of K_0.

Let $F(C)$ denote the mod 2 boundary of the chain C. Now $F(K_0)$ may have vertices incident with more than two 1-cells of $F(K_0)$. Let m be the number of such vertices, and P one such vertex. On each edge of the star of P introduce a new vertex, then introduce a new vertex in each 2-cell of the star and subdivide the cell by joining this vertex to the vertices on its boundary. Add to K_0 the 2-cells of this subdivision having P as a vertex and not already in K_0. The so enlarged K_0 has now only $m - 1$ singular points on its boundary. Continuation of this process leads to a K_0 whose boundary has no singular points and is therefore the sum of a set $\gamma_1 + \cdots + \gamma_k$ of pairwise disjoint simple closed curves.

Suppose the cycle $\gamma = \gamma_{i_1} + \cdots + \gamma_{i_p}$ is a proper part of $F(K_0)$ which bounds in $\bar{K}$. Then $\bar{K} = C_1 + C_2$ where C_1 is a finite chain having γ as boundary. Now $K_0 - \gamma$ is a connected set; for if it were the sum of separate pieces $A_1 + A_2$, $F(K_0)$ would be the sum $F(A_1) + F(A_2)$. As K_0 is connected the boundaries of A_1 and A_2 would meet; but this contradicts the fact that K_0 has no singular points on its boundary. Therefore K_0 lies in C_1 or in C_2.

Case 1. Suppose K_0 is in C_1. In this case, let $K_0' = C_1$ and we obtain an enlarged K_0 satisfying all conditions satisfied by K_0. The only point needing verification is that K_0' is connected. But this follows readily from the fact that K_0 is connected and $F(K_0')$ is part of $F(K_0)$.

Case 2. Suppose K_0 is in C_2. In this case, let $K_0' = K_0 + C_1$. Obviously K_0' is connected.

Thus in either case we obtain a complex K_0' containing K_0, satisfying all conditions satisfied by K_0, and having fewer closed curves on its boundary. If some proper part of $F(K_0')$ bounded in $\bar{K}$, repetition of this argument leads to a K_0'' with still fewer closed curves on its boundary. We are therefore eventually lead to a complex $K \supset K_0$ such that no proper part of $F(K)$ bounds in $\bar{K}$. Let $\Gamma_1, \cdots, \Gamma_s$ be the closed curves of $F(K)$.

Let K_1 be a finite subcomplex of $\bar{K}$ having K in its interior. We can, by repeating the foregoing argument, insure that K_1 satisfies the same conditions as K. Let $\Delta_1, \cdots, \Delta_t$ be the closed curves of $F(K_1)$. Let H be the mod 2 sum of K and K_1. Then $F(H) = \sum_1^s \Gamma_i + \sum_1^t \Delta_j$. Let H' be a component of H; then $F(H') \subset F(H)$. Now $F(H')$ is not $\sum \Delta_j$, for then $K_1 - H'$ would be a manifold. It is not $\sum \Gamma_i$, for then $H' + K$ would be a manifold. It cannot be a proper part of $\sum \Gamma_i$ nor of $\sum \Delta_j$ for no proper part of either cycle bounds. Thus $F(H') = \gamma_1 + \gamma_2$ where $\gamma_1 \subset \sum \Gamma_i$ and $\gamma_2 \subset \sum \Delta_j$.

Suppose, as is impossible, that γ_1 contains Γ_i and $\Gamma_j (i \neq j)$. Let T be an

arc of H' joining Γ_i and Γ_j, and T' an arc of K joining the same two points of Γ_i and Γ_j. Then $\Gamma = T + T'$ is a cycle of $\bar{K}$. As K contains a basis of $\bar{K}$, Γ is homologous to a cycle Γ' of K. Suppose C is a chain such that $F(C) = \Gamma + \Gamma'$. Let C' be the part of C on H'. Then $F(C') = T +$ a subset of γ_1. A closed curve Γ'' of $F(C')$ containing T would be such that $\Gamma'' - T$ formed an arc of $F(K)$ joining Γ_i and Γ_j. But $F(K)$ consists of disjoint closed curves, so this is impossible. Hence γ_1 is just one of the curves $\Gamma_1, \cdots, \Gamma_s$.

Suppose, as is impossible, that γ_2 contains Δ_i and $\Delta_j (i \neq j)$. As K contains a basis of $\bar{K}$, Δ_i is homologous to a cycle of K and therefore to a cycle on $F(K)$. Suppose C is a chain of $\bar{K}$ not in K whose boundary is Δ_i plus part of $F(K)$. If C had a part not in K_1, that part would have a part of $F(K_1)$ as boundary. As this is impossible C is in K_1. Since C has Δ_i on its boundary it has in common with H' at least some 2-cells adjacent to Δ_i. But H' is connected and does not cross $F(C)$. Hence C contains H'. On the other hand the component C_0 of C containing H' does not cross $F(H')$. Hence $H' \supset C_0$. Thus $H' = C_0$. But $F(C_0)$ is Δ_i plus part of $F(K)$. This means that Δ_j is not in $F(H')$ which is a contradiction.

Thus each component of H has for boundary just one curve from each of $\Gamma_1, \cdots, \Gamma_s$ and $\Delta_1, \cdots, \Delta_t$. Thus $t = s$ and we may renumber the Δ's and the components $H_1^1, \cdots, H_s^1$ of H so that $F(H_i^1) = \Gamma_i + \Delta_i$.

Any 1-cycle on H_i^1 is homologous to a cycle of K. It therefore bounds on H_i^1 or is homologous to Δ_i. It follows that H_i^1 is a cylinder. To see this, introduce 2-cells having Γ_i, Δ_i respectively as boundaries. The result is a 2-manifold with zero mod 2 Betti number. It is therefore a sphere; so H_i^1 is a part of a sphere lying between two closed curves.

To complete the proof we construct in this way a sequence $K_1, K_2, \cdots$ whose sum is $\bar{K}$ so that $K_n - K_{n-1}$ is a set of s disjoint cylinders $H_1^n, \cdots, H_s^n$. Finally we set $H_i = \sum_{n=1}^{\infty} H_i^n$ and the lemma is proved.

LEMMA 2. *If M is a compact 2-manifold without boundary, and g is a continuum on M with zero* mod 2 *1-dimensional Betti number, then in each open set about g there is a closed 2-cell having g in its interior.*

Let D be a component of $M - g$ and suppose D is decomposed in the manner of Lemma 1. The statement "g has zero Betti number" means that, if K_1 is any complex of M containing g, there is a smaller complex K_2 of M containing g such that each of its 1-cycles bounds in K_1. The cycle Γ_1, by means of the cylinder H_1, is homologous to a cycle of K_2. Hence Γ_1 bounds in M; it therefore separates M. But this means that D has but the one cylinder H_1 of approach to g.

Should $M - g$ have another component D' we could choose K_1 so as not to contain all of D'. Choose K_1 so as not to meet K. As seen above, Γ_1 bounds in $K_1 + H_1$. This chain together with K would form a non-zero 2-cycle lying on a proper part of M. As this is impossible, $M - g$ has only the one component D.

It follows that Γ_1 divides M into K and $H + g$. Thus $H + g$ is a 2-manifold with the boundary Γ_1. Choose K_1 to lie in $H + g$. As H is a cylinder any 1-cycle on $H + g$ is homologous to a cycle of K_2 and therefore bounds in K_1. Thus every 1-cycle of $H + g$ bounds in $H + g$. It follows that $H + g$ is a 2-cell. By sectioning H with closed curves parallel to Γ_1 we readily obtain a decreasing sequence of 2-cells whose intersection is g. This proves the lemma.

II. Upper Semi-Continuous Collections of Continua on a Two-Manifold

Throughout the following theorems M will denote a compact connected 2-dimensional manifold without boundary (it need not be orientable), G will denote an upper semi-continuous[3] collection of continua filling M, g will denote an element of G, $R(S)$ will denote the mod 2 1-dimensional Betti number of the set S. A *generalized cactoid* is a continuous curve whose every maximal cyclic element[4] is a 2-manifold and only a finite number of these are different from 2-spheres. It is well known that G may be made into a topological space[5] in such a way that G is the continuous image of M under the monotone transformation which sends the point $P \in M$ into the element $g \in G$ containing P. It follows that G is a continuous curve.

Theorem 1. *If G contains at least two elements and $R(g) = 0$ for each $g \in G$, then G is homeomorphic to M.*

Theorem 2. *If, for each $g \in G$, $M - g$ is connected and has just one cylinder of approach to g (see Lemma 1), then G is a compact 2-manifold without boundary, $R(G) \leqq R(M)$, and if M is orientable so is G.*

Theorem 3. *If, for each $g \in G$, each component of $M - g$ has just one cylinder of approach to g, then G is a generalized cactoid, and $R(G) \leqq R(M)$. Conversely, if C is a generalized cactoid, there exists an M and a G such that G is homeomorphic to C and $R(G) = R(M)$.*

Theorem 4. *The conditions of Theorems 1, 2 and 3 are necessary.*

Theorem 5. *With no restriction on G, there exists a finite number of spaces $C_0, C_1, \cdots, C_k$ where C_0 is a generalized cactoid and C_i is obtained by identifying just two points of $C_{i-1} (i = 1, 2, \cdots, k)$ such that G is homeomorphic to C_k and $R(G) \leqq R(M) - k$. Conversely, given $C_0, C_1, \cdots, C_k$ as above, there exists an M and a G such that G is homeomorphic to C_k and $R(G) = R(M) - k$.*

Corollary. *The class of continuous images of compact 2-dimensional manifolds without boundary under monotone transformations is composed of just those*

---

[3] A collection G of mutually exclusive closed sets is *upper semi-continuous* if, for each $g \in G$ and each open set $U \supset g$, there exists an open set $V \supset g$ such that every $g' \in G$ which meets V lies in U.

[4] For the theory of the cyclic elements of a continuous curve see G. T. Whyburn, *Concerning the structure of a continuous curve*, Amer. Jour. of Math., vol. 50 (1928), pp. 167–194.

[5] A subset S of G is open if, for each $g \in S$, there exists an open set $U \supset g$ in M such that every $g' \in G$ meeting U is in S. For further details see R. L. Moore, *Concerning upper semi-continuous collections of continua*, Trans. Amer. Math. Soc., vol. 27 (1925), pp. 416–428, or see [1].

continuous curves each of which can be obtained from a generalized cactoid by making a finite number of identifications.

PROOF OF THEOREM 1. Let $g \in G$ be arbitrary. By Lemma 2 there exists a closed 2-cell C on M whose interior contains g. Since G is upper semi-continuous, there exists an open set $V \supset g$ such that every $g' \in G$ meeting $\bar{V}$ lies in the interior of C. Let S be a 2-sphere obtained by introducing a new 2-cell C' having exactly its boundary in common with C. Let G_0 be the collection of continua filling S defined as follows: each element of G meeting $\bar{V}$ is an element of G_0; each point of S not in such a continuum is an element of G_0. It is not difficult to prove that G_0 is upper semi-continuous. Since for each $g_0 \in G_0$, $R(g_0) = 0$, no g_0 can separate S. Hence by a theorem of R. L. Moore,[6] G_0 is a 2-sphere. Let W be the set of continua of G lying wholly in V. The set W is common to G and G_0, and is open in both spaces. Since G_0 is a sphere, it follows that G is locally a plane at the element g. Hence G is a compact 2-manifold.

To show that M and G are homeomorphic we now need to prove only that M and G have the same 1-dimensional homology groups. But this follows directly from a theorem of Vietoris[7] and the fact that each element of G has zero Betti number for the dimensions zero and one.

PROOF OF THEOREM 2. If G satisfies the condition of Theorem 1, Theorem 2 is immediate. Suppose this is not so; then we shall construct a new manifold M_1 and collection G_1 filling M_1 so that

(1) G_1 and G are homeomorphic,

(2) $R(M_1) < R(M)$.

By assumption there is an element of G, say g_0, such that $R(g_0) \neq 0$. By Lemma 1 and the hypothesis of Theorem 2, there is a connected complex K in $M - g_0$ whose boundary is a simple closed curve Γ, and $M - K - g_0$ is an open cylinder H. Let M_1 be the manifold obtained by adding a 2-cell E to K having Γ as boundary.

To verify (2), observe first that $R(M_1) = R(K)$. We must show that $R(K) < R(M)$. Since the homology group of K is a subgroup of that of M (see proof of Lemma 1), we need only exhibit a cycle of M not homologous to one in K. Now $K' = g_0 + H + \Gamma$ is a connected 2-manifold with boundary Γ. It cannot be a 2-cell; for, since H is a cylinder, the contrary would mean that g_0 is the intersection of a decreasing sequence of 2-cells and therefore $R(g_0) = 0$. Since K' is not a 2-cell, it contains a non-bounding 1-cycle Γ'. If Γ'' were a 1-cycle of K such that $\Gamma' + \Gamma''$ bounds a chain C in M, the part C' of C on K' would have $\Gamma' + \Gamma$ as boundary. As Γ bounds K' this would mean that Γ' bounds in K'. This proves (2).

To construct G_1 we define a mapping f of M into M_1 as follows. It shall be

---

[6] See reference in 5.

[7] L. Vietoris, *Über den höheren Zusammenhang kompakter Raume und eine Klasse von zusammenhangstreuen Abbildungen*, Math. Ann., vol. 97 (1927), pp. 454–472. In particular see (8b) p. 470.

the identity over K. Let A be a point in the interior of E and set $f(g_0) = A$. Finally let f map H homeomorphically onto $E - A$. This latter map can clearly be chosen so that f is continuous over M. Thus, under f, G images into an upper semi-continuous collection G_1 filling M_1, and G_1 is homeomorphic with G.

If M_1, G_1 does not satisfy Theorem 1, there is a $g_1 \in G_1$ such that $R(g_1) \neq 0$. We then construct M_2, G_2 satisfying relative to M_1, G_1 the conditions analogous to (1) and (2). Due to (2) this process continues for a number of steps $= k \leqq R(M)$. Then M_k, G_k satisfy the conditions of Theorem 1. It follows that G_k, and therefore G, is homeomorphic to M_k.

Proof of First Part of Theorem 3. We first show that each maximal cyclic element G_1 of G is a 2-manifold. To this end we define a function $f(g)$ for all $g \in G_1$. If g does not separate M, $f(g) = g$. If g separates M, $f(g) = g +$ the sum $D(g)$ of all components of $M - g$ excepting the one containing the connected set[8] $G_1 - g$.

It is clear that $f(g)$ is a continuum. Furthermore if g, $g' \in G_1$ are distinct, then $f(g)$ and $f(g')$ do not meet. To see this, we note that $D(g')$ by definition does not meet g; hence $f(g')$ does not meet g. But $f(g')$ is connected; it therefore lies in the component of $M - g$ omitted from $D(g)$.

Each point P of M is in some $f(g)$. If P is not in G_1, let U be the component of $M - G_1$ containing P. Since G_1 is a maximal cyclic element, each component of $G - G_1$ has just one limit point in G_1. This means that the boundary of U lies in just one continuum $g \in G_1$. Then P is in $f(g)$.

Thus the set $\{f(g)\}$ for all $g \in G_1$ is a collection G' of mutually exclusive continua filling M. It is upper semi-continuous. To see this, suppose the contrary. We then obtain a sequence $\{f(g_n)\}$, points P_n, $Q_n \in f(g_n)$ such that $\{P_n\}$ converges to $P \in f(g)$ and $\{Q_n\}$ converges to $Q \in f(g')$. Since $D(g)$ and $D(g')$ are open sets not meeting any $f(g_n)$, it follows that $P \in g$ and $Q \in g'$. Let U be any connected open set about g. It then contains some P_n. If $P_n \in D(g_n)$, as g is not in $D(g_n)$, and U is connected, it follows that U contains a point $P_n' \in g_n$. Similarly an open connected set $V \supset g'$ contains a point $Q_n' \in g_n$. But the existence of $\{P_n'\}$, $\{Q_n'\}$ converging to P, Q contradicts the upper semi-continuity of the collection G. Hence G' is upper semi-continuous.

The correspondence $f(g)$ between G_1 and G' is certainly $1 - 1$. It is continuous; for, if $\{g_n\}$ converges to g, then the sequence of continua $\{f(g_n)\}$ must have limit points in $f(g)$. As G' is upper semi-continuous, it follows that $\{f(g_n)\}$ converges to $f(g)$. As G_1 and G' are compact spaces, they must be homeomorphic under the function f.

Finally let us observe that G' satisfies the conditions of Theorem 2. By construction $M - f(g)$ is connected and is a component of $M - g$. By the

[8] If G' denotes a subset of the space G we use the same symbol G' to denote the point set on M obtained by adding together all elements of G'.

hypothesis of Theorem 3, this component has just one cylinder of approach to g. It follows that G', and therefore G_1, is a 2-manifold.

By a theorem of Vietoris,[7] $R(G) \leqq R(M)$. By a theorem of Whyburn,[9] the Betti number of G is the sum of the Betti numbers of its maximal cyclic elements. As $R(G)$ is finite, at most a finite number of cyclic elements can have a Betti number > 0. All other maximal cyclic elements must therefore be 2-spheres. Thus G is a generalized cactoid, and the proof of the first part of Theorem 3 is complete.

PROOF OF CONVERSE PART OF THEOREM 3. We must prove that any generalized cactoid C is the monotone image of a 2-manifold with the same Betti number. We shall not give complete details but shall let the proof rest on the argument given by Moore[1] to show that any cactoid is a monotone image of a 2-sphere. In Moore's argument a fundamental lemma for the 2-sphere is stated and used without proof. One cannot therefore check his proof to see that the analogous lemma for a 2-manifold holds. Since the lemma is in no sense trivial, we shall state and prove it in complete detail.

LEMMA 3. *If M is a compact 2-manifold without boundary and $\{P_n\}$ is any countable sequence of points in M, then there exists in M a countable set of non-overlapping closed 2-cells $\{W_n\}$ whose diameters tend to zero and there exists a continuous transformation T of M onto itself such that $T(W_n) = P_n$ and T is a homeomorphism between $M - \sum W_n$ and $M - \sum P_n$.*

We shall define closed 2-cells $W_1, W_2, \cdots$ and continuous transformations $\Pi_0, \Pi_1, \Pi_2, \cdots$ of M onto M. We then define $T_0, T_1, \cdots$ by the formulas $T_0(x) = \Pi_0(x)$, $T_n(x) = \Pi_n(T_{n-1}(x))$. If x denotes a point of M we let x^n denote $T_n(x)$. We want the following properties to hold, where σ is the metric on M:

1. $\sigma(x, \Pi_n(x)) < 1/2^n$.

2. If x and y are points of M such that $\sigma(x, y) \geqq 1/n$ then $\sigma(x^n, y^n) \geqq (1 - 1/2^{n+1}) \cdot \sigma(x^{n-1}, y^{n-1})$.

3. $W_1, W_2, \cdots$ are mutually exclusive closed 2-cells and diameter of $W_n < 1/n$.

4. $T_n(W_k) = P_k$ if $k \leqq n$, but otherwise T_n is $1 - 1$.

The definitions are given by induction. Let $T_0 = \Pi_0 =$ the identity on M. Suppose $W_1, W_2, \cdots, W_{n-1}$; $\Pi_0, \Pi_1, \cdots, \Pi_{n-1}$ have been defined so that 1–4 hold true. Let U_n be an open 2-cell containing $T_{n-1}^{-1}(P_n)$ such that $\bar{U}_n$ does not intersect $\sum_1^{n-1} W_i$ and the diameter of $U_n < 1/n$. Let $X_n = T_{n-1}(U_n)$, and let ϵ_n be a positive number < 1 such that $S(P_n, 2\epsilon_n)$ (i.e., the set of points with distance from $P_n < 2\epsilon_n$) lies in X_n. (Such an ϵ_n exists since clearly T_{n-1} is a homeomorphism over $\bar{U}_n$. Let Y_n and Z_n be open 2-cells such that $P_n \subset Z_n$, $\bar{Z}_n \subset Y_n$, and diameter $Y_n < \epsilon_n/2^{n+1}$. Let Π_n be now defined as a transformation of M onto M which is continuous everywhere, is $1 - 1$ except that it

[9] G. T. Whyburn, *Cyclic elements of higher order*, Amer. Jour. of Math., vol. 56 (1934), pp. 133–146. In particular see (2.5) p. 138.

maps all of $\overline{Z}_n$ into P_n, and is the identity transformation on $M - Y_n$. Then Π_n maps $Y_n - \overline{Z}_n$ homeomorphically onto $Y_n - P_n$. Let $V_n = T_{n-1}^{-1}(Y_n)$ and $W_n = T_{n-1}^{-1}(\overline{Z}_n)$.

We now show that properties 1–4 hold for the sets $W_1, W_2, \cdots, W_n$ and transformations $\Pi_0, \Pi_1, \cdots, \Pi_n$.

1. If x is in Y_n we have $\sigma(x, \Pi_n(x)) \leqq$ diameter $Y_n < \epsilon_n/2^{n+1} < 1/2^n$. If x is not in Y_n then $\Pi_n(x) = x$.

2. If neither x nor y is in V_n then Π_n is the identity on x^{n-1} and on y^{n-1}, whence 2 holds. Suppose x is in V_n. Then if $\sigma(x, y) \geqq 1/n$ it follows that y is in $M - U_n$. Now $\sigma(x^{n-1}, x^n) \leqq$ diameter of $Y_n \leqq \epsilon_n/2^{n+1}$, while $y^n = y^{n-1}$. Thus from the triangle axiom we get $\sigma(x^n, y^n) = \sigma(x^n, y^{n-1}) \geqq \sigma(x^{n-1}, y^{n-1}) - \sigma(x^{n-1}, x^n) \geqq \sigma(x^{n-1}, y^{n-1}) - \epsilon_n/2^{n+1}$. Now $\sigma(x^{n-1}, y^{n-1}) \geqq \sigma(P_n, y^{n-1}) - \sigma(P_n, x^{n-1}) > 2\epsilon_n - \epsilon_n/2^{n+1} > \epsilon_n$. Using this we get $\sigma(x^n, y^n) > (1 - 1/2^{n+1}) \cdot \sigma(x^{n-1}, y^{n-1})$.

3. The closed set W_n is in the open set U_n which does not intersect W_k with $k < n$.

4. Now $T^n(W_k) = \Pi_n(T_{n-1}(W_k))$. If $k < n$ then $T_{n-1}(W_k) = P_k$, and $\Pi_n(P_k) = P_k$. If $k = n$, $T_{n-1}(W_k) = \overline{Z}_k$, and $\Pi_n(\overline{Z}_k) = P_k$. Thus if $k \leqq n$, $T_n(W_k) = P_k$. If Q is distinct from $P_1, P_2, \cdots, P_n$, then $\Pi_n^{-1}(Q)$ is a single point, as is also $T_{n-1}^{-1} \cdot \Pi_n^{-1}(Q)$.

Thus we have an infinite sequence $W_1, W_2, \cdots$ of mutually exclusive closed 2-cells, and a uniformly convergent sequence of continuous transformations $T_0, T_1, \cdots$ of M onto M such that properties 1–4 hold. Define T as follows: $T(x) = \lim_{n=\infty} T_n(x)$. Then clearly T is a continuous transformation of M onto M and, for each i, $T(W_i) = P_i$. Suppose x and y are distinct points of M. Determine n so that $\sigma(x, y) \geqq 1/n$. If $x^n = y^n$ then x and y are in some one of the sets $W_1, W_2, \cdots, W_n$. If $x^n \neq y^n$ then, by property 2,

$$\sigma(T(x), T(y)) \geqq \sigma(x^n, y^n) \prod_{k=2}^{\infty} (1 - 1/2^{n+k}) > \tfrac{1}{2}\sigma(x^n, y^n) > 0.$$

This completes the proof of lemma 3.

We return now to the proof that a generalized cactoid C is a monotone image of a 2-manifold. We consider first the special case where but one maximal cyclic element M of C is different from a 2-sphere. The cut points of C belonging to M form a countable set; let these be arranged in a sequence $\{P_n\}$. Let M' be a replica of M. We now apply Lemma 3 where the 2-cells $\{W_n\}$ are in M'. We define a collection G of continua in M' as follows. The boundary Γ_n of W_n for each n shall be an element. Every point not on any Γ_n shall be an element. The transformation T applied to the points of M' not in the interior of any W_n maps the corresponding subset of G homeomorphically into M. It follows that G is upper semi-continuous. The image of W_n in G is a sphere S_n attached to M at the point P_n. Let R_n be P_n plus all components of $C - M$ having P_n as limit. Since M is the only maximal cyclic element of C not a 2-

sphere, it follows that R_n is a cactoid. According to the theorem of Moore there exists a monotone transformation of S_n onto R_n. This transformation can be so chosen as to leave P_n fixed; for, if it mapped P_n' into P_n, we could then apply a homeomorphism of S_n into itself carrying P_n into P_n'. If this transformation of S_n onto R_n is effected for each n, the result is a monotone transformation of M' onto C. We note that the Betti number of M' equals that of C.

Suppose now that C is a generalized cactoid with just two maximal cyclic elements not 2-spheres. We can then express C as a sum $C_1 + C_2$ of two generalized cactoids with just one point P in common and such that each has just one maximal cyclic element not a 2-sphere. By what has just been proved there are manifolds M_1, M_2 with the Betti numbers of C_1, C_2 and monotone transformations $T_1(M_1) = C_1$, $T_2(M_2) = C_2$. Let g_1, g_2 be the inverse images of P in M_1, M_2. Choose a point P_1 in g_1 and let S_1 be a 2-cell about P_1 with polar coordinates ρ, $\theta(0 \leqq \rho \leqq 1)$ with P_1 as origin. Let f_1 map $S_1 - P_1$ on the ring $\frac{1}{2} < \rho \leqq 1$ by the transformation $\theta' = \theta$, $\rho' = (\rho + 1)/2$, and let f_1 be the identity on $M_1 - S_1$. If S_1' is the open 2-cell $0 \leqq \rho < \frac{1}{2}$ and Γ_1 is its boundary, the result is a homeomorphism of $M_1 - P_1$ onto $M_1 - (S_1' + \Gamma_1)$. We obtain a collection C_1' of continua in $M_1 - S_1'$ by taking the image of each continuum of the collection C_1 in M_1 except that we add Γ_1 to the image of $g_1 - P_1$. It is clear that C_1' is homeomorphic to C_1 and Γ_1 corresponds to P. Perform the analogous operation in C_2. Now join $M_1 - S_1'$ and $M_2 - S_2'$ by a cylinder H having $\Gamma_1 + \Gamma_2$ as its boundary. In the resulting manifold M we obtain a collection G of continua by taking as elements the continua of $M_1 - S_1'$ and $M_2 - S_2'$ excepting the ones containing Γ_1 and Γ_2. These two are added together with H to form a new continuum. The transformation $T_1 f_1^{-1}$ and $T_2 f_2^{-1}$ define a monotone transformation T of $M - H$ on C except that $T^{-1}(P)$ is composed of the two pieces $f_1 T_1^{-1}(P) + \Gamma_1$ and $f_2 T_1^{-1}(P) + \Gamma_2$. If we set $T(H) = P$, then T becomes a true monotone transformation of M on C.

The Euler number $E(M)$ of a complex is $\alpha_0 - \alpha_1 + \alpha_2$ where α_i = number of i-cells. Removing S_1' from M_1 decreased its Euler number by 1: $E(M_1 - S_1') = E(M_1) - 1$, $E(M_2 - S_2') = E(M_2) - 1$. Now $E(H) = 0$ since H is a cylinder. Hence $E(M) = E(M_1 - S_1') + E(M_2 - S_2') = E(M_1) + E(M_2) - 2$. But for a 2-manifold M without boundary, $E(M) = 2 - R(M)$. Substituting in this last relation gives $R(M) = R(M_1) + R(M_2)$. Thus $R(M) = R(C_1) + R(C_2) = R(C)$.

The argument just given for the case where C has just two maximal cyclic elements not 2-spheres lends itself to an obvious induction which provides a proof of our proposition.

PROOF OF THEOREM 4. If G is a generalized cactoid then no point of a maximal cyclic element is a local cut point of the cyclic element. Suppose, as is impossible, that $M - g$ has a component D with at least two cylinders of approach to g. The image G_0 of $g + D$ in G is a cyclic chain of G and g is a non-cut point of G_0. Since D has at least two cylinders of approach to g, it follows that g

is a local cut point of G_0. Hence g is not an end point of G_0. It therefore lies on a maximal cyclic element of G and is a local cut point of that cyclic element. This contradiction proves the necessity of the condition of Theorem 3.

If G is a manifold, then surely it is a generalized cactoid. It has just been shown that each component of $M - g$ has just one cylinder of approach to g. But G can have no cut point, hence $M - g$ is connected. Thus the condition of Theorem 2 is necessary.

To see the necessity of the condition of Theorem 1, suppose G is a manifold and $g \in G$ is such that $R(g) \neq 0$. From the first assumption it follows that the conditions of Theorem 2 are satisfied. This second assumption was made at the beginning of the proof of Theorem 2. We then constructed a new manifold M_1 and collection G_1 filling M_1 such that G_1 and G were homeomorphic and $R(M_1) < R(M)$. It follows that $R(G_1) < R(M)$. Thus $R(g) \neq 0$ leads necessarily to $R(G) < R(M)$.

Proof of First Part of Theorem 5. If G satisfies the condition of Theorem 3, Theorem 5 is immediate. Suppose G does not satisfy the condition of Theorem 3; then we shall construct a new manifold M_1 and collection G_1 on M_1 satisfying

(1) $R(M_1) = R(M) - 2$,

(2) G can be obtained by identifying two points of G_1.

If Theorem 3 does not apply to M_1, G_1, we construct M_2, G_2 in an analogous manner; and so on. Due to (1) this process can be continued for a number of steps $= k \leqq R(M)/2$. Then Theorem 3 must apply to M_k, G_k. Setting $C_i = G_{k-i} (i = 0, 1, \cdots, k-1)$ and $C_k = G$, the first statement of Theorem 5 is found to be proved. For the second, observe first that $R(M_k) = R(M) - 2k$. By Theorem 3, $R(G_k) \leqq R(M_k)$. Identifying two points of a connected space increases its R by 1. Hence $R(G) = R(G_k) + k$. Combining these three relations gives $R(G) \leqq R(M) - k$. The construction of M_1, G_1 follows.

As G does not satisfy the condition of Theorem 3, there is a $g_0 \in G$ and a component D of $M - g_0$ with two cylinders of approach to g_0. Let H be one of the cylinders of D bounded by the closed curve Γ (see Lemma 1). Cut M at Γ denoting the two lips of the cut by Γ_1, Γ_2 where Γ_1 is next to H. Introduce two 2-cells E_1, E_2 having Γ_1, Γ_2 respectively as boundaries. The resulting manifold is M_1. It is connected; for, since D has another cylinder of approach to g_0, cutting at Γ does not separate M. We verify (1) as follows. The Euler number of a 2-complex is $\alpha_0 - \alpha_1 + \alpha_2$ where α_i = number of i-cells. Cutting at Γ increases α_0 and α_1 by the same amount. Adding E_1 and E_2 increases α_2 by 2. Hence the Euler numbers of M, M_1 are related by $E(M_1) = E(M) + 2$. As M, M_1 are 2-manifolds without boundary $E(M) = 2 - R(M)$ and $E(M_1) = 2 - R(M_1)$. The last three relations imply (1).

We construct G_1 by means of a mapping f of M on M_1. Observe first that there is given by construction a correspondence ϕ between M and $M_1 - (E_1 + E_2)$ which is two-valued over Γ. Let $f = \phi$ over $M - (H + \Gamma)$. Let f map Γ into Γ_2 in the same manner as ϕ. Let A be a point of E_2 and let f map H homeomorphically onto $E_2 - A$. This is done so that f is continuous at Γ.

Then f is $1-1$ between M and $M_1 - [\phi(H) + \Gamma_1 + E_1 + A]$. It is discontinuous only at g_0 and there only with respect to sequences approaching g_0 along H. The inverse of f is continuous wherever defined.

Since f is continuous over each $g \in G$, $f(g)$ is a continuum in M_1. Furthermore $f(g_1)$ and $f(g_2)$ do not meet if g_1 and g_2 are distinct. The elements of G_1 shall be, first, the point A, next, the continua $f(g)$ for all $g \neq g_0$, and finally the continuum $g_0^* = \phi(g_0 + H) + \Gamma_1 + E_1$. Clearly G_1 fills M_1.

If G_1 were not upper semi-continuous there would exist a sequence $\{\bar{g}_n\}$ in G_1, points P_n, $Q_n \in \bar{g}_n$ such that $\{P_n\}$ and $\{Q_n\}$ converged to points P, Q in different continua $\bar{g}$, $\bar{g}'$ of G_1. We can clearly suppose that $\bar{g}_n$ is an $f(g)$, say $f(g_n) = \bar{g}_n$. By passing to a subsequence we can obtain that $\{g_n\}$ converges in G to an element g. If $g \neq g_0$, then from the continuity of f we must have that $f(g_n)$ converges to $f(g)$. As this is contrary to assumption, $\{g_n\}$ must converge to g_0. By passing to a subsequence we can insure that either $\{g_n\}$ converges to g_0 along H or no g_n meets H. In the first case $f(g_n)$ converges to A, in the second, to g_0^*. Thus in any case we have a contradiction. So G_1 is upper semi-continuous.

To prove (2), let ψ map G_1 in G as follows: for each $g \neq g_0$, let $\psi f(g) = g$, let $\psi(A) = g_0$, and $\psi(g_0^*) = g_0$. We need prove only that ψ is continuous. Suppose $\{f(g_n)\}$ converges to $\bar{g} \in G_1$. If $\bar{g}$ is an $f(g)$ then, since f is a homeomorphism in some open set about g, $g_n \to g$. If $\bar{g} = A$, then $\{g_n\}$ converges to g_0 along H. Suppose $\bar{g} = g_0^*$. In this case $g_n \to g_0$; for, if a subsequence of $\{g_n\}$ converged to $g' \neq g_0$, we could infer from the continuity of f at g' that the corresponding subsequence of $\{f(g_n)\}$ converged to $f(g')$. This proves that ψ is continuous and completes the proof of the first part of Theorem 5.

Proof of the Converse Part of Theorem 5. The case $k = 0$ is contained in Theorem 3. Suppose the proposition is true for all $k \leqq h$, and let C_0, C_1, $\cdots$ C_h, C_{h+1} be a sequence as in the theorem. By assumption there is a 2-manifold M and collection G in M so that G is homeomorphic with C_h and $R(G) \leqq R(M) - h$. Let g_1, g_2 be the two points of G which are identified in forming C_{h+1}. Choose points $P_1 \in g_1$, $P_2 \in g_2$ and let S_1, S_2 be non-overlapping 2-cell neighborhoods of P_1, P_2 respectively. We suppose in S_1 we have polar coordinates (r, θ), $0 \leqq r < 1$, $0 \leqq \theta < 2\pi$. The transformation $\theta' = \theta$, $r' = (r+1)/2$ applied to S_1 maps $S_1 - P_1$ into the annular ring $\frac{1}{2} < r < 1$. Perform a similar transformation on $S_2 - P_2$. The result is a homeomorphism T of $M - (P_1 + P_2)$ into $M - (C_1 + C_2)$, where C_1, C_2 are closed 2-cells in S_1, S_2 respectively. Remove from M the interiors of C_1, C_2 and introduce a cylinder H whose boundary consists of the two closed curves forming the boundaries of C_1 and C_2. This operation changes M into a new manifold M'. The collection G' in M' shall consist of the elements $T(g)$ for each $g \in G$ which is not g_1 or g_2 and of the additional element $g^* = H + T(g_1 - P_1) + T(g_2 - P_2)$. It is trivial to verify that G' is upper semi-continuous and that it is the result of identifying the two points g_1, g_2 of G. Hence G' is homeomorphic with C_{h+1}. Removing the two 2-cells from M decreased its Euler number by 2. The Euler number of H is

zero. Hence $E(M') = E(M) - 2$. Transforming to Betti numbers gives $R(M') = R(M) + 2$. Since identifying two points increased R by 1, we have $R(G') = R(G) + 1$. As $R(G) = R(M) - h$, we obtain $R(G') = R(M') - (h + 1)$. This completes the proof.

CONCLUSION. Our results lead in a fairly obvious way[10] to an answer to the question: What is the space G if M is a compact manifold with a finite number of closed curves as a boundary? The following two paragraphs suggest extensions of our work which are not so obvious.

(1) What is the space G if M is (a) a non-compact manifold, (b) an arbitrary 2-complex, or (c) a Peano space whose maximal cyclic elements have prescribed structures such as: being 2-complexes.

(2) If M is a 3-manifold, in particular a 3-sphere, then it is known that the monotone image G of M can contain topologically *any* Peano curve. This question arises: What conditions on the elements of G, as continua on the 3-sphere M, are necessary and sufficient that G be homeomorphic to M? In particular, is this condition sufficient: If g is any element of G then $M - g$ is an open 3-cell?[11]

PRINCETON UNIVERSITY.

[10] One can fit 2-cells onto the bounding curves so as to obtain a manifold M' without boundary. A collection G' filling M' can then be defined, whose elements are (1) the elements of G, and (2) the individual points of the added 2-cells. Our results characterize G'. To get G from G' we need only omit a finite number of open 2-cells, namely the images of the 2-cells added to M.

[11] Compare James F. Wardwell, *Continuous transformations preserving all topological properties*, Amer. Jour. of Math., vol. 58 (1936), pp. 709–726.

ANNALS OF MATHEMATICS
Vol. 39, No. 4, October, 1938

CONFORMAL GEOMETRY ON A SURFACE[1]

BY GEORGE COMENETZ

(Received May 31, 1938)

Introduction

This paper generalizes to the case of curved surfaces certain theorems of plane conformal geometry due to Kasner.[2] The results obtained are: expressions for the invariants of horn angles of first and second order contact and the general right angle, and the first few conformal symmetry formulas. Conceivably the invariants and the formulas which Kasner found for the plane might have carried over with no change whatever to surfaces, but the fact proves to be more interesting than that. In the cases of lowest order there is actually no change, but for higher order the Gaussian curvature of the surface makes its appearance.

1. The invariants

The term "surface" will be used for an oriented, analytic surface element (that is, element of a two-dimensional Riemannian space). "Curve" will mean an oriented, analytic curve element.

Let C and D be two curves which lie on a surface Σ and pass through a point O of Σ. We denote the geodesic curvature of C at O by Γ, the value at O of the first derivative of curvature with respect to arc-length by Γ_1, the value of the second derivative by Γ_2, and so on. The similar quantities for D are denoted by $\Delta, \Delta_1, \Delta_2, \cdots$. For the Gaussian curvature of Σ at O we write K.

First, suppose that C and D pass through O in the same direction. The curvilinear angle which they form is then known as a "horn angle" with vertex at O. If $\Delta - \Gamma \neq 0$, the horn angle is said to be of first order contact. If $\Delta - \Gamma = 0$ but $\Delta_1 - \Gamma_1 \neq 0$, the horn angle is of second order contact.

Let the surface Σ be mapped by a direct conformal transformation onto any other surface. A horn angle of first or second order contact on Σ goes into a horn angle of the same order of contact on the new surface. Of course the curvatures and their derivatives at the new vertex may be changed in value. Our first group of theorems, on the other hand, presents certain combinations of these quantities whose values do not change.

---

[1] Abstracts in Bulletin of the American Mathematical Society, vol. 42 (1936), p. 806.

[2] *Conformal Geometry*, Proceedings of the Fifth International Congress of Mathematicians, vol. 2 (1912), p. 81; *Geometry of Conformal Symmetry (Schwarzian Reflection)*, Annals of Mathematics, vol. 38 (1937), p. 873; *The Two Conformal Invariants of Fifth Order*, Transactions of the American Mathematical Society, vol. 44 (1938), p. 25.

THEOREM I: *If a horn angle of first order contact is drawn on a surface, and the surface is mapped by a direct conformal transformation onto any other surface, the value of "Kasner's invariant"*

$$\frac{\Delta_1 - \Gamma_1}{(\Delta - \Gamma)^2}, \tag{1}$$

and the algebraic sign of $\Delta - \Gamma$, *remain unaltered.*[3]

Kasner's invariant is probably the only one in the present series whose form is independent of whether the surfaces under consideration are curved or flat. For the next simplest conformal invariant, given in the following theorem, already involves the Gaussian curvature K.

THEOREM II: *If a horn angle of second order contact is drawn on a surface, and the surface is mapped by a direct conformal transformation onto any other surface, the value of*

$$\frac{4(\Delta_1 - \Gamma_1)(\Delta_3 - \Gamma_3) - 5(\Delta_2 - \Gamma_2)^2 - 4(\Gamma^2 + K)(\Delta_1 - \Gamma_1)^2}{(\Delta_1 - \Gamma_1)^3}, \tag{2}$$

and the algebraic sign of $\Delta_1 - \Gamma_1$, *remain invariant.*

The Γ which occurs in (2) is the common initial curvature of the two sides of the horn angle, and might equally well be written Δ. In view of the presence of K it would be proper to speak of (2) as a simultaneous invariant of the horn angle and the surface. The above result and those which follow all reduce, as they should, to Kasner's results in the plane if we set the Gaussian curvature equal to zero.

Next, let the angle from C to D at O be 90°, and furthermore let $\Delta_1 + \Gamma_1$ be different from zero. The curvilinear angle composed of C and D is then known as a "general right angle." The conformal invariant of a general right angle depends not only on the value of the Gaussian curvature at the vertex, but also on its rates of change. To express the latter we introduce a net of curves.

Let an arbitrary orthogonal net of curves be drawn on the surface Σ, such that the two curves of the net which pass through O have the directions of C and D at that point. Let α be the curvature at O of the net curve tangent to C, and β that of the net curve tangent to D. Arc-length along the family of curves of the net which includes the curve tangent to C is denoted by σ, and arc-length along the other family of the net by τ. The symbols K_σ, K_τ, $K_{\sigma\tau}$, $K_{\tau\sigma}$ stand for the values at O of the indicated first and second partial derivatives of the Gaussian curvature, the differentiation being performed with respect to arc-length along the curves of the net. As is well known, such differentiation need not be commutative; that is, $K_{\sigma\tau}$ and $K_{\tau\sigma}$ need not be equal.

[3] G. Comenetz, *Kasner's Invariant and Trihornometry*, American Mathematical Monthly, vol. 45 (1938), p. 82. The fact, first surmised by Kasner, that (1) applies also on a surface was observed independently by V. Hlavatý in a review in the Zentralblatt, vol. 14 (1936), p. 178.

A right angle of course remains a right angle after a conformal transformation, and in fact a general right angle remains general. This is implied in the following statement.

THEOREM III: *If a general right angle is drawn on a surface, and the surface is mapped by a direct conformal transformation onto any other surface, the value of*

$$(3)\quad \frac{1}{(\Delta_1+\Gamma_1)^2}[\Delta_3-\Gamma_3-5(\Gamma\Delta_2+\Gamma_2\Delta)-\Delta^2\Delta_1+5\Gamma^2\Delta_1-5\Gamma_1\Delta^2+\Gamma^2\Gamma_1 - K(\Delta_1-\Gamma_1)+2(\Delta K_\tau-\Gamma K_\sigma)+\tfrac{1}{2}(\alpha K_\sigma-\beta K_\tau+K_{\sigma\tau}+K_{\tau\sigma})],$$

and the algebraic sign of $\Delta_1+\Gamma_1$, *remain invariant.*

The values of K_σ and K_τ really depend only on the point O and the initial directions of C and D, not on the particular net curves which pass through O in those directions. The quantity $\alpha K_\sigma-\beta K_\tau+K_{\sigma\tau}+K_{\tau\sigma}$ likewise depends only on O and the two orthogonal directions of C and D at O, not on the particular orthogonal net of curves.

There is good reason for believing that the expressions in theorems I–III are the unique conformal invariants of the two kinds of horn angles and of the right angle respectively (see §5). If this is so, one numerical value and one algebraic sign are a complete set of distinguishing marks for a horn angle or a right angle of the above kinds within the domain of conformal geometry.

2. Symmetry formulas

Let C now be thought of as a fixed curve on Σ, and D as an arbitrary curve on Σ cutting C at the fixed point O. The angle θ from C to D at O may have any value. The symmetry formulas deal with the effect upon D when it is reflected by conformal symmetry across the "base curve" C.

The *conformal symmetry* or *Schwarzian reflection* determined by C may be defined as the anti-conformal transformation of Σ into itself which leaves every point of C fixed. An equivalent definition is this: let a portion of a plane P be mapped conformally onto the surface Σ in such a way that some straight line segment L in P goes into the curve C on Σ. (That this is possible follows easily from the known fact that any analytic curve in a plane is conformally equivalent to a straight line.) Then the mapping transforms the ordinary reflection in P across the line L, into the conformal symmetry on Σ across the base curve C. A symmetry is of period two, for its square is directly conformal and leaves each point of C fixed, and hence is the identity. The symmetry in C is unique, since if there were two such, their product would have to be the identity by the previous argument. Each being of period two, they would have to be the same.

Let the curve which is the image of D under the symmetry in C be called $\bar{D}$. As a symmetry is anti-conformal, $\bar{D}$ cuts C at O at the angle $-\theta$. Represent the values at O of curvature and its arc-length derivatives for $\bar{D}$ by $\bar{\Delta}$, $\bar{\Delta}_1$, $\bar{\Delta}_2, \cdots$. The symmetry formulas express the curvature and its derivatives

for $\bar{D}$ in terms of the corresponding quantities for C and D (namely $\Gamma, \Gamma_1, \cdots$ and $\Delta, \Delta_1, \cdots$ respectively).

THEOREM IV: *The first three symmetry formulas are:*

$$(4)\quad \bar{\Delta} + \Delta = 2\Gamma \cos\theta,$$

$$(5)\quad \bar{\Delta}_1 + \Delta_1 = 2\Gamma_1 \cos 2\theta,$$

$$(6)\quad \bar{\Delta}_2 + \Delta_2 = 2\Gamma_2 \cos 3\theta - 4\Delta\Gamma_1 \sin 2\theta + 4\Gamma(\Gamma_1 + \Delta_1)\sin\theta + K_\tau \sin\theta \sin 2\theta.$$

The fourth formula has not been found except in two special cases: when D is tangent to C and when D is perpendicular to C.

THEOREM IV′: *The fourth symmetry formula is, in the special cases* $\theta = 0°$ *and* $\theta = 90°$ *respectively,*

$$(7)\quad \bar{\Delta}_3 + \Delta_3 = 2\Gamma_3 - 4\Gamma_1(\Gamma - \Delta)(\Gamma - 2\Delta) - 4\Delta_1\Gamma(\Gamma - \Delta),$$

$$(8)\quad \bar{\Delta}_3 + \Delta_3 = 2\Gamma_3 + 10(\Delta\Gamma_2 + \Delta_2\Gamma) - 4(3\Gamma^2\Gamma_1 - 2\Delta^2\Gamma_1 + 5\Delta_1\Gamma^2) - 4(\Gamma_1 K + \Gamma K_\sigma - \Delta K_\tau) - (\alpha K_\sigma - \beta K_\tau + K_{\sigma\sigma} + K_{\tau\sigma}).$$

The purpose of the symmetry formulas might be described as follows: the successive formulas enable us to construct with greater and greater precision, in a directly geometric way, the image of any given curve after a Schwarzian reflection. It would be desirable to have an immediate geometrical construction by which, if a point and the base curve are given, the reflected point can be found. However, so far as the author is aware, essentially the only such construction known is for the special case where the base curve is a circle in the plane. The Schwarzian reflection then reduces to ordinary inversion, and the immediate geometrical construction is that by reciprocal radii vectores.

3. Transformation laws for curvature

The proofs below, if considered for the plane case, differ from Kasner's proofs in that curvilinear coördinates are used and the conformal transformations are represented by identity of coördinates instead of by power series.

It is sufficient to prove theorems I–III when one of the two surfaces is a plane, since a surface-to-plane followed by a plane-to-surface mapping is a general surface-to-surface mapping.

Consider therefore a conformal mapping between a plane P and the surface Σ. Let u, v be rectangular cartesian coördinates in P, and let them be carried onto Σ as cúrvilinear coördinates by the mapping. The element of arc ds on the plane is defined by

$$(9)\qquad ds^2 = du^2 + dv^2,$$

while for the element of arc dS on the surface we will have

$$(10)\qquad dS^2 = e^{2n}(du^2 + dv^2),$$

where $n(u, v)$ is some point function on the surface. The coördinates on Σ are isothermal because the mapping is conformal.

Let the two curves and the point on P whose images on Σ are C, D and O, be called c, d and o respectively. Let curvature and its arc-length derivatives at o be denoted by γ, γ_1, $\cdots$ for c, and by δ, δ_1, $\cdots$ for d. The curvilinear angle c, d is mapped conformally onto the curvilinear angle C, D, and what we have to show is that the various expressions in theorems I–III have the same value when formed for one angle as for the other.

The fundamental formulas in all of this work are the laws of transformation of geodesic curvature and its arc-length derivatives under conformal mapping. We shall state as many of these as we need, for the curve C at the point O; of course they will apply in a similar way to any other point of C or of any other curve.

On the surface Σ, let φ be the angle at O from the coördinate line $v = const.$ which passes through O, to the curve C. In view of the conformality, this is the same as the angle on P from the straight line $v = const.$ through o, to the curve c.

If we substitute from (10) in a standard formula for geodesic curvature[4] and use the fact that $\cos\varphi = e^n du/dS$ and $\sin\varphi = e^n dv/dS$, we find that

$$e^n\Gamma = \gamma + n_u \sin\varphi - n_v \cos\varphi. \tag{11}$$

This is the transformation law of curvature.

The various quantities in (11) are evaluated at O (or at o), but as remarked above (11) holds similarly at each point of C (or c). Hence we may differentiate (11) with respect to arc-length along C. In performing this and the following differentiations it is convenient to keep in mind the relations

$$\frac{d}{dS} = e^{-n}\frac{d}{ds}, \qquad \frac{d\varphi}{dS} = e^{-n}\gamma, \qquad \frac{dz}{dS} = e^{-n}(z_u \cos\varphi + z_v \sin\varphi),$$

the z in the third relation being any function of u and v. Differentiating (11) with respect to S and then substituting for Γ from (11), we obtain

$$e^{2n}\Gamma_1 = \gamma_1 + \tfrac{1}{2}(n_{uu} - n_{vv} - n_u^2 + n_v^2)\sin 2\varphi - (n_{uv} - n_u n_v)\cos 2\varphi. \tag{12}$$

This is the transformation law of the first derivative of curvature.

In exactly the same way, by differentiation of (12) with respect to S, the transformation law of the second derivative is found to be

$$\begin{aligned} e^{3n}\Gamma_2 = {} & \gamma_2 - 2\gamma_1(n_u\cos\varphi + n_v\sin\varphi) + \gamma[(n_{uu} - n_{vv} - n_u^2 + n_v^2)\cos 2\varphi \\ & + 2(n_{uv} - n_u n_v)\sin 2\varphi] - (n_{uuv} - n_v n_{uu} + 2n_u^2 n_v)\cos 3\varphi \\ & - (n_{uvv} - n_u n_{vv} + 2n_u n_v^2)\sin 3\varphi \\ & + (n_{uuu} + n_{uvv} - 4n_u n_{uu} + 8n_v n_{uv} + 2n_u^3 + 2n_u n_v^2)\cos^2\varphi\sin\varphi \\ & - (n_{uuv} + n_{vvv} + 8n_u n_{uv} - 4n_v n_{vv} + 2n_u^2 n_v + 2n_v^3)\cos\varphi\sin^2\varphi \\ & + 3n_{uv}(n_u\cos\varphi - n_v\sin\varphi). \end{aligned} \tag{13}$$

[4] Beltrami, Opere I, p. 178.

The third derivative will be needed only when $\varphi = 0°$ and when $\varphi = 90°$. Differentiating (13) and setting $\varphi = 0°$ as we do so, we arrive at

$$(14)\quad e^{4n}\Gamma_3 = \gamma_3 - 5n_u\gamma_2 - 2n_v\gamma\gamma_1 - \gamma_1(n_{uu} + n_{vv} - 5n_u^2 - n_v^2) + 4\gamma^2(n_{uv} - n_un_v)$$
$$+ \gamma(2n_{uuu} - 3n_{uvv} - 9n_un_{uu} + 7n_vn_{uv} + 6n_un_{vv} + 5n_u^3 - 7n_un_v^2)$$
$$- (n_{uuuv} - n_vn_{uuu} - 6n_un_{uuv} - 4n_{uu}n_{uv} + 7n_un_vn_{uu} + 11n_u^2n_{uv} - 6n_u^3n_v).$$

For $\varphi = 90°$ we have

(15) [same as (14), but with sign reversed for each derivative of n whose subscripts include an odd number of v's, and with u and v subscripts then interchanged].

4. Derivation of the invariants

Suppose that C and D form a horn angle of first order contact. Then the angle φ is the same for both, and $\Delta - \Gamma \neq 0$. For C we have (11), and for D the corresponding formula would be

$$(16)\qquad e^n\Delta = \delta + n_u \sin\varphi - n_v\cos\varphi.$$

By subtraction of (11) from (16),

$$(17)\qquad e^n(\Delta - \Gamma) = \delta - \gamma.$$

As e^n is positive, this shows that the algebraic sign of the difference of curvature remains unaltered.

In exactly the same way, from formula (12) we obtain

$$(18)\qquad e^{2n}(\Delta_1 - \Gamma_1) = \delta_1 - \gamma_1 .$$

Squaring (17) and dividing into (18), we have

$$(19)\qquad \frac{\Delta_1 - \Gamma_1}{(\Delta - \Gamma)^2} = \frac{\delta_1 - \gamma_1}{(\delta - \gamma)^2}.$$

This means that (1) is invariant. The proof of theorem I is therefore complete.

Suppose next that C and D form a horn angle of second order contact. The angle φ is still the same for both, but now $\Delta = \Gamma$ and $\Delta_1 - \Gamma_1 \neq 0$.

The rectangular cartesian coördinates u, v in the plane P were chosen at random, and were then used to describe the given conformal mapping. We are at liberty to assume now that the choice of coördinates is so made that the line $v = const.$ has the direction of the curve c at o. The angle φ is therefore $0°$ for C and c in all that follows.

In the present case $\varphi = 0°$ for D also. Let $\varphi = 0°$ in (11)–(14), and let each of these equations be written down for C and for D. Together with $\Gamma = \Delta$ they are a set of nine equations. The process of finding an invariant consists of eliminating n, since it is n that defines the particular conformal mapping.

The attempt to eliminate n fails however: after use has been made of all nine

equations, the Laplacian of n still remains. But the Laplacian is practically the Gaussian curvature K of Σ, for if we substitute from (10) in a formula for Gaussian curvature, we find that

$$K = -e^{-2n}(n_{uu} + n_{vv}). \tag{20}$$

It is now easy to show that (2) is equal to the corresponding expression in the small letters, where K in the latter expression is replaced by zero. This is as it should be, for the Gaussian curvature of the plane P is zero. Hence (2) is a conformal invariant.

The algebraic sign of $\Delta_1 - \Gamma_1$ is seen to be invariant as before. This completes the proof of theorem II.

Finally, let C and D form a general right angle. Then $\varphi = 0°$ for C and $\varphi = 90°$ for D. By substituting in (11)–(15) we obtain a set of eight equations, but it is not possible to eliminate n from these, even with the aid of (20).

Further auxiliary relations may be introduced, however. For if we differentiate (20) with respect to arc-length σ along the curve $v = const.$, we have

$$e^{3n}K_\sigma = 2n_u L - L_u\,, \tag{21}$$

where L stands for $n_{uu} + n_{vv}$. Similarly the derivative K_τ along the curve $u = const.$ is given by

$$e^{3n}K_\tau = 2n_v L - L_v\,. \tag{22}$$

Differentiation of (21) with respect to τ yields the cross-derivative $K_{\sigma\tau}$:

$$e^{4n}K_{\sigma\tau} = 2L(n_{uv} - 3n_u n_v) + 3n_v L_u + 2n_u L_v - L_{uv}\,.$$

A similar relation for $K_{\tau\sigma}$ is derived from (22).

By (11), the curvatures α and β of the curves $v = const.$ and $u = const.$ at O are given respectively by

$$e^n\alpha = -n_v\,, \qquad e^n\beta = n_u\,,$$

since the curvatures of the corresponding straight lines $u = const.$ and $v = const.$ in P are zero.

It may now be verified that (3) is equal to the corresponding expression formed in the plane P. Theorem III is not yet proved, though, because we have used a particular net of curves (the parameter lines) in evaluating α and β and the derivatives of the Gaussian curvature, whereas theorem III speaks of *any* orthogonal net tangent to the given right angle.

In the first place, (21) and (22) hold for any curves having the directions of C and D respectively, at O. Hence K_σ and K_τ are in reality functions of the point O and the directions of C and D at O only, and would have the same values no matter what orthogonal net tangent to the right angle were drawn on Σ.

As for the quantity

$$\alpha K_\sigma - \beta K_\tau + K_{\sigma\tau} + K_{\tau\sigma}\,, \tag{23}$$

the symbols in it other than K_σ and K_τ do depend on the particular net, but the value of the whole expression does not. So far as isothermal nets are concerned, this could be shown by a very simple argument from what has already been proved, using the definition of an isothermal net as the conformal image of an orthogonal net of straight lines in a plane, and the fact that (3) is invariant under conformal transformations within one plane.

For a general orthogonal net, however, a separate calculation is necessary. Let $dv/du = f(u, v)$ represent any family of curves on Σ such that the curve through O is tangent to C there; that is, such that f vanishes at O. Then $du/dv = -f$ represents the orthogonal trajectories. Taking σ, τ, α and β as referring to this new net, it is easy to compute that at O

$$e^n K_\sigma = K_u\,, \qquad e^{2n} K_{\sigma\tau} = K_{uv} - n_v K_u + f_v K_v\,, \qquad e^n \alpha = -n_v + f_u\,,$$

with similar relations for K_τ, $K_{\tau\sigma}$ and β. Forming the expression (23) we find that f cancels. Hence (23) is in fact invariant under any change of the orthogonal reference net.

The last remark holds good even if K in (23) stands for any scalar whatever on the surface Σ, since the argument did not depend on K's being the Gaussian curvature. *For any scalar K, the value of* (23) *is the same for all orthogonal nets having given directions at O.*

It can be shown in a similar way that among all nets of curves which pass through O in the given directions (but are not necessarily orthogonal except at O), those for which (23) has the same value as it has for the orthogonal nets are characterized by this property: for either family of curves in the net, if the orthogonal trajectory through O is drawn, it has contact of higher than first order with the other family's curve through O.

5. Uniqueness of the invariants

Let two horn angles of first order contact in a plane have the same value for (1) and the same sign for the curvature difference. Given any positive integer n, there exists a conformal transformation in the plane which makes the sides of one angle coincide with the corresponding sides of the other to within an order of contact higher than n. Otherwise expressed, in a plane *a horn angle of first order contact has no conformal invariant of finite order other than Kasner's invariant and the sign of the curvature difference.*[5]

It follows easily that the italicized statement is true also for horn angles on surfaces. For if we map such angles onto a fixed plane, the value of (1) and the sign do not change, and then we can apply the above result in that plane.

It is probable, though not yet proved, that two horn angles in a plane which have the same value of (1) and like signs are conformally equivalent without any reservation. If so, the same would be true on surfaces.

Statements of exactly the same kind can be made for horn angles of second order contact and for general right angles.

[5] See Kasner's 1912 paper.

6. Derivation of the symmetry formulas

The conformal symmetry across the curve C on the surface Σ is determined by a conformal mapping of a straight line L in a plane P onto C, as explained in §2. We may suppose that the rectangular cartesian coördinates u, v in P are chosen so that L is the u-axis. Then C is the $v = 0$ line on Σ.

Given the symmetric curves D and $\bar{D}$ on Σ, we know that the two curves d and $\bar{d}$ in P which correspond to them are mirror images of each other under the ordinary reflection in L. Consequently the initial curvature $\bar{\delta}$ of $\bar{d}$ is the negative of the initial curvature δ of d, and similarly $\bar{\delta}_1 = -\delta_1$, $\bar{\delta}_2 = -\delta_2$, $\cdots$.

Applying formula (11) to the curves d and D, we have

$$e^n \Delta = \delta + n_u \sin\theta - n_v \cos\theta, \tag{24}$$

since the angle from C to D was called θ. For $\bar{d}$ and $\bar{D}$ the angle is $-\theta$:

$$e^n \bar{\Delta} = \bar{\delta} - n_u \sin\theta - n_v \cos\theta. \tag{25}$$

Finally for L and C,

$$e^n \Gamma = -n_v, \tag{26}$$

as the curvature of L is zero and the initial angle is zero.

Adding (24) and (25), using the fact that $\delta + \bar{\delta} = 0$, and substituting from (26) we obtain the first symmetry formula (4).

The second symmetry formula (5) follows from (12) precisely as (4) from (11). For the third formula (6) we use (13), but then also (24), (25), (26), the three corresponding relations derived from (12), and finally (22). For (7) and (8) a similar procedure is followed. The remarks about (23) of course apply to the last term in (8).

In his papers on fifth order invariants and symmetry in the plane, Kasner employed the various formulas to derive a number of further results. Some of these, such as the remarkably simple symmetry formulas for a curve tangent to the base curve, the derivation of the right angle invariant from the invariant of a horn angle of second order contact by means of conformal symmetry, the relative invariants, and at least part of the discussion on bisection, carry over at once to surfaces. As for inversion, it would naturally be defined on a general surface, at any rate in the present connection, as *Schwarzian reflection in a geodesic circle* (of constant geodesic curvature). Inversion formulas could then be derived from the symmetry formulas by setting all the derivatives of curvature for the base curve equal to zero. But on a surface not of constant Gaussian curvature, this kind of inversion would not have the property which it has in the plane of turning (geodesic) circles into circles. For if we set $\Gamma_1 = \Gamma_2 = \Delta_1 = \Delta_2 = 0$ in (6), it does not follow that $\bar{\Delta}_2$ must vanish.

ST. JOHN'S COLLEGE,
ANNAPOLIS, MD.

the symbols in it other than K_σ and K_τ do depend on the particular net, but the value of the whole expression does not. So far as isothermal nets are concerned, this could be shown by a very simple argument from what has already been proved, using the definition of an isothermal net as the conformal image of an orthogonal net of straight lines in a plane, and the fact that (3) is invariant under conformal transformations within one plane.

For a general orthogonal net, however, a separate calculation is necessary. Let $dv/du = f(u, v)$ represent any family of curves on Σ such that the curve through O is tangent to C there; that is, such that f vanishes at O. Then $du/dv = -f$ represents the orthogonal trajectories. Taking σ, τ, α and β as referring to this new net, it is easy to compute that at O

$$e^n K_\sigma = K_u\,, \qquad e^{2n} K_{\sigma\tau} = K_{uv} - n_v K_u + f_v K_v\,, \qquad e^n \alpha = -n_v + f_u\,,$$

with similar relations for K_τ, $K_{\tau\sigma}$ and β. Forming the expression (23) we find that f cancels. Hence (23) is in fact invariant under any change of the orthogonal reference net.

The last remark holds good even if K in (23) stands for any scalar whatever on the surface Σ, since the argument did not depend on K's being the Gaussian curvature. *For any scalar K, the value of* (23) *is the same for all orthogonal nets having given directions at O.*

It can be shown in a similar way that among all nets of curves which pass through O in the given directions (but are not necessarily orthogonal except at O), those for which (23) has the same value as it has for the orthogonal nets are characterized by this property: for either family of curves in the net, if the orthogonal trajectory through O is drawn, it has contact of higher than first order with the other family's curve through O.

5. Uniqueness of the invariants

Let two horn angles of first order contact in a plane have the same value for (1) and the same sign for the curvature difference. Given any positive integer n, there exists a conformal transformation in the plane which makes the sides of one angle coincide with the corresponding sides of the other to within an order of contact higher than n. Otherwise expressed, in a plane *a horn angle of first order contact has no conformal invariant of finite order other than Kasner's invariant and the sign of the curvature difference.*[5]

It follows easily that the italicized statement is true also for horn angles on surfaces. For if we map such angles onto a fixed plane, the value of (1) and the sign do not change, and then we can apply the above result in that plane.

It is probable, though not yet proved, that two horn angles in a plane which have the same value of (1) and like signs are conformally equivalent without any reservation. If so, the same would be true on surfaces.

Statements of exactly the same kind can be made for horn angles of second order contact and for general right angles.

[5] See Kasner's 1912 paper.

6. Derivation of the symmetry formulas

The conformal symmetry across the curve C on the surface Σ is determined by a conformal mapping of a straight line L in a plane P onto C, as explained in §2. We may suppose that the rectangular cartesian coördinates u, v in P are chosen so that L is the u-axis. Then C is the $v = 0$ line on Σ.

Given the symmetric curves D and $\bar{D}$ on Σ, we know that the two curves d and $\bar{d}$ in P which correspond to them are mirror images of each other under the ordinary reflection in L. Consequently the initial curvature $\bar{\delta}$ of $\bar{d}$ is the negative of the initial curvature δ of d, and similarly $\bar{\delta}_1 = -\delta_1$, $\bar{\delta}_2 = -\delta_2$, $\cdots$.

Applying formula (11) to the curves d and D, we have

$$e^n \Delta = \delta + n_u \sin\theta - n_v \cos\theta, \tag{24}$$

since the angle from C to D was called θ. For $\bar{d}$ and $\bar{D}$ the angle is $-\theta$:

$$e^n \bar{\Delta} = \bar{\delta} - n_u \sin\theta - n_v \cos\theta. \tag{25}$$

Finally for L and C,

$$e^n \Gamma = -n_v, \tag{26}$$

as the curvature of L is zero and the initial angle is zero.

Adding (24) and (25), using the fact that $\delta + \bar{\delta} = 0$, and substituting from (26) we obtain the first symmetry formula (4).

The second symmetry formula (5) follows from (12) precisely as (4) from (11). For the third formula (6) we use (13), but then also (24), (25), (26), the three corresponding relations derived from (12), and finally (22). For (7) and (8) a similar procedure is followed. The remarks about (23) of course apply to the last term in (8).

In his papers on fifth order invariants and symmetry in the plane, Kasner employed the various formulas to derive a number of further results. Some of these, such as the remarkably simple symmetry formulas for a curve tangent to the base curve, the derivation of the right angle invariant from the invariant of a horn angle of second order contact by means of conformal symmetry, the relative invariants, and at least part of the discussion on bisection, carry over at once to surfaces. As for inversion, it would naturally be defined on a general surface, at any rate in the present connection, as *Schwarzian reflection in a geodesic circle* (of constant geodesic curvature). Inversion formulas could then be derived from the symmetry formulas by setting all the derivatives of curvature for the base curve equal to zero. But on a surface not of constant Gaussian curvature, this kind of inversion would not have the property which it has in the plane of turning (geodesic) circles into circles. For if we set $\Gamma_1 = \Gamma_2 = \Delta_1 = \Delta_2 = 0$ in (6), it does not follow that $\bar{\Delta}_2$ must vanish.

ST. JOHN'S COLLEGE,
ANNAPOLIS, MD.

ON THE PERMUTATION OF THE CONVERGENTS OF A CONTINUED FRACTION AND RELATED CONVERGENCE CRITERIA

By John Q. Jordan and Walter Leighton, Jr.

(Received May 31, 1938)

1. Introduction

Let

$$1 + \overset{\infty}{\underset{n=1}{K}} (x_n/1) = 1 + \frac{x_1}{1} + \frac{x_2}{1} + \cdots \tag{1.1}$$

be a continued fraction in which the x_n are complex numbers different from zero. Pringsheim[1] has shown that the conditions

$$|x_n| \leqq 1/4 \qquad (n = 2, 3, 4, \cdots), \tag{1.2}$$

are sufficient for the convergence of (1.1). Szasz[2] has pointed out that $\frac{1}{4}$ is the largest number which can be used as the right hand member of (1.2).

Leighton and Wall[3] showed that sufficient conditions for the convergence of (1.1) are

$$|x_{2n+2}| \geqq 25/4, \quad |x_{2n+1}| \leqq 1/4 \ (n = 1, 2, 3, \cdots), \quad |x_2| \geqq 5. \tag{1.3}$$

Denoting by X_n and Y_n the numerator and denominator respectively of the n^{th} convergent of (1.1) they found explicit expressions for the elements of a continued fraction with its n^{th} convergent A_n/B_n satisfying the conditions

$$\begin{aligned} A_n &= \alpha_n X_n + \beta_n X_{n+1}, \\ B_n &= \alpha_n Y_n + \beta_n Y_{n+1}, \end{aligned} \qquad (n = 0, 1, 2, \cdots), \tag{1.4}$$

where the α_n, β_n are arbitrary complex numbers.

If the numbers α_n and β_n are assigned the values

$$\begin{aligned} \alpha_{2n} &= 0, & \beta_{2n} &= 1, \\ \alpha_{2n+1} &= 1, & \beta_{2n+1} &= 0, \end{aligned}$$

the transformation interchanges consecutive pairs of convergents of (1.1). The two continued fractions then converge or diverge together, and it is possible to establish, among other results, conditions (1.3).

---

[1] O. Perron, "Die Lehre von den Kettenbrüchen," 1913 edition, pp. 254–259.

[2] O. Szasz, "Über die Erhaltung der Konvergenz Kettenbrüche bei independenter Veränderlichkeit aller ihrer Elementer," Crelle, Bd. 147 (1916), pp. 132–160.

[3] Walter Leighton and H. S. Wall, "On the Transformation and Convergence of Continued Fractions," American Journal of Mathematics, Vol. LVIII, No. 2, April, 1936.

The present paper investigates the transformations of the form (1.4) which result from permuting sets of three and four consecutive convergents. New forms of convergence criteria appear. The discussion is divided into three parts. In section 2 we carry out in some detail a typical case where the convergents are permuted in sets of four. In section 3 we tabulate the results of the other cases which arise in permuting four consecutive convergents of (1.1). In section 4 we tabulate the corresponding results when permutations are made in sets of three. In order to simplify the statement of our results we shall use the following abbreviated notation. Let

$$b_m + \overset{\infty}{\underset{n=m+1}{K}} (a_n/b_n) \tag{1.5}$$

be a continued fraction formed from

$$b_0 + \overset{\infty}{\underset{n=1}{K}} (a_n/b_n) \tag{1.6}$$

by dropping all those terms preceding b_m. We denote the n^{th} convergent of (1.5) by $A_{n,m}/B_{n,m}$.[4] In particular we shall find the following formulae useful.

$$\begin{aligned}
X_{-1,m} &= 1, \\
X_{0,m} &= 1, \\
X_{1,m} &= 1 + x_{m+1}, \\
X_{2,m} &= 1 + x_{m+1} + x_{m+2}, \\
X_{3,m} &= 1 + x_{m+1} + x_{m+2} + x_{m+3} + x_{m+1}x_{m+3}, \\
X_{4,m} &= 1 + x_{m+1} + x_{m+2} + x_{m+3} + x_{m+4} + x_{m+1}x_{m+3} \\
&\qquad + x_{m+1}x_{m+4} + x_{m+2}x_{m+4}, \\
X_{5,m} &= 1 + x_{m+1} + x_{m+2} + x_{m+3} + x_{m+4} + x_{m+5} + x_{m+1}x_{m+5} \\
&\quad + x_{m+2}x_{m+5} + x_{m+3}x_{m+5} + x_{m+1}x_{m+3}x_{m+5} + x_{m+1}x_{m+3} \\
&\qquad + x_{m+1}x_{m+4} + x_{m+2}x_{m+4}.
\end{aligned} \tag{1.7}$$

In the above the $X_{n,m}$'s denote numerators of convergents of a continued fraction $1 + \overset{\infty}{\underset{n=m+1}{K}} (x_n/1)$ of the type (1.5) and formed from (1.1).

2. A typical transformation

We shall first prove an important and typical theorem and corollary.

THEOREM 2.1: *If the numbers x_n satisfy the conditions*

$$\left|\frac{x_{4n}}{X_{1,4n-2}X_{1,4n}}\right| \leqq 1/4, \quad \left|\frac{x_{2n+1}}{X_{1,2n}}\right| \leqq 1/4, \quad \left|\frac{1}{x_{4n-2}}\right| \leqq 1/4, \quad (n = 1, 2, 3, \cdots), \tag{2.1}$$

the continued fraction (1.1) *converges.*

[4] Perron, ibid., p. 14.

COROLLARY 2.1: *If for* $n = 1, 2, 3, \cdots$ *the numbers* x_n *satisfy the conditions*

$$|x_{2n+1}| \leqq 1/5, \quad |x_{4n-2}| \geqq 4, \quad |x_{4n}| \leqq 1/5 \tag{2.2}$$

the continued fraction (1.1) *converges.*

If X_n and Y_n are determined from the relations

$$\begin{aligned} X_{-1} &= 1, \quad Y_{-1} = 0, \quad X_0 = 1, \quad Y_0 = 1, \\ X_n &= X_{n-1} + x_n X_{n-2}, \\ Y_n &= Y_{n-1} + x_n Y_{n-2}, \end{aligned} \qquad (n = 1, 2, 3, \cdots),$$

the quotient X_n/Y_n $(n = 0, 1, 2, \cdots)$ is defined[5] to be the n^{th} convergent of (1.1). Similarly let the n^{th} convergent of the continued fraction

$$b_0 + \overset{\infty}{\underset{n=1}{K}} (a_n/b_n) \tag{2.3}$$

be designated by A_n/B_n, where

$$\begin{aligned} A_{-1} &= 1, \quad B_{-1} = 0, \quad A_0 = b_0, \quad B_0 = 1, \\ A_n &= b_n A_{n-1} + a_n A_{n-2}, \\ B_n &= b_n B_{n-1} + a_n B_{n-2}, \end{aligned} \qquad (n = 1, 2, 3, \cdots). \tag{2.4}$$

We seek to determine the numbers $b_0, b_1, b_2, \cdots, a_1, a_2, \cdots$ in terms of the numbers x_n in such a way that

$$\begin{aligned} A_{4n} &= X_{4n+1}, \\ A_{4n+1} &= X_{4n}, \\ A_{4n+2} &= X_{4n+2}, \\ A_{4n+3} &= X_{4n+3}, \\ B_{4n} &= Y_{4n+1}, \\ B_{4n+1} &= Y_{4n}, \\ B_{4n+2} &= Y_{4n+2}, \\ B_{4n+3} &= Y_{4n+3}, \end{aligned} \qquad (n = 0, 1, 2, \cdots). \tag{2.5}$$

For simplicity we designate relations (2.5) by the matrix

$$\begin{Vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix}. \tag{2.5a}$$

[5] Perron, ibid., p. 5.

By the substitution in (2.3) of the values of the A's and B's given by (2.5) we arrive at

$$\begin{aligned} X_{4n+1} &= b_{4n}X_{4n-1} + a_{4n}X_{4n-2}, \\ Y_{4n+1} &= b_{4n}Y_{4n-1} + a_{4n}Y_{4n-2}. \end{aligned} \tag{2.6}$$

But from (1.1) and the recursion relation

$$\begin{aligned} X_{4n+1} &= X_{4n} + x_{4n+1}X_{4n-1} = (1 + x_{4n+1})X_{4n-1} + x_{4n}X_{4n-2}, \\ Y_{4n+1} &= Y_{4n} + x_{4n+1}Y_{4n-1} = (1 + x_{4n+1})Y_{4n-1} + x_{4n}Y_{4n-2}. \end{aligned} \tag{2.7}$$

By subtracting (2.7) from the corresponding equations of (2.6) we have

$$\begin{aligned} [b_{4n} - (1 + x_{4n+1})]X_{4n-1} + [a_{4n} - x_{4n}]X_{4n-2} &= 0 \\ [b_{4n} - (1 + x_{4n+1})]Y_{4n-1} + [a_{4n} - x_n]Y_{4n-2} &= 0. \end{aligned} \tag{2.8}$$

Hence $a_{4n} = x_{4n}$, $b_{4n} = 1 + x_{4n+1}$, since

$$\begin{vmatrix} X_{4n-1} & X_{4n-2} \\ Y_{4n-1} & Y_{4n-2} \end{vmatrix} = (-1)^{4n-1}x_{4n-1}x_{4n-2}\cdots x_1 \neq 0,$$

and by similar procedure we obtain as complete results

$$\begin{aligned} a_{4n} &= x_{4n}, \quad a_{4n+1} = -x_{4n+1}, \quad a_{4n+2} = 1, \quad a_{4n+3} = -x_{4n+2}x_{4n+3}, \\ b_{4n} &= 1 + x_{4n+1}, \; b_{4n+1} = 1, \; b_{4n+2} = x_{4n+2}, \; b_{4n+3} = 1 + x_{4n+3}. \end{aligned} \tag{2.9}$$

In general the initial conditions, that is the values of b_0, b_1, and a_1 have to be determined separately since always $A_{-1} = X_{-1} = 1$, $B_{-1} = Y_{-1} = 0$, but in this particular case it is unnecessary, as is evident from the relations (2.5), or (2.5a). Application of Pringsheim's condition for the convergence of (2.3)[6] to (2.9) yields the theorem.

The corollary follows immediately from this.

THEOREM 2.2. *If the quantities x_n are functions of any set of variables (finite or infinite in number), the continued fraction* (1.1) *converges uniformly in every closed region throughout which conditions* (2.1), *or* (2.2) *are satisfied.*

The proof is immediate.

In passing it is well to emphasize that the results (2.9), or the left-hand members of the expressions (2.1) without the absolute value signs, are important not only because they yield new convergence criteria but also because they define a transformation of the continued fraction (1.1) into an equivalent one (i.e. one which converges or diverges with (1.1), and if both converge, then they converge to the same value).

3. Further results of permuting convergents four at a time

In the following the matrices indicate the particular transformation which yields the given results. We have the theorem.

[6] Perron, ibid., p. 259.

THEOREM 3.1. *The continued fraction* (1.1) *converges if*

$$|\alpha_n| \leqq 1/4 \qquad (n = 2, 3, 4, \cdots),$$

where the value of α_n *is given by any one of the following sets of relations.*

(3.1) $\begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{Vmatrix}$

$$\alpha_{4n} = 1/x_{4n}, \qquad \alpha_{4n+2} = \frac{x_{4n+2}}{X_{1,4n}X_{1,4n+2}},$$

$$\alpha_{2n+1} = -\frac{x_{2n+1}}{X_{1,2n}}, \ (n = 1, 2, 3, \cdots), \qquad \alpha_2 = \frac{x_2}{X_{1,2}}.$$

(3.10) $\begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix}$

$$\alpha_{4n-1} = 1/x_{4n-1}, \qquad \alpha_{2n} = -\frac{x_{2n}}{X_{1,2n-1}},$$

$$\alpha_{4n+1} = \frac{x_{4n+1}}{X_{1,4n-1}X_{1,4n+1}}, \ (n = 1, 2, 3, \cdots).$$

(3.2) $\begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{Vmatrix}$

$$\alpha_{4n-1} = -X_{1,4n-2}, \qquad \alpha_{4n} = -X_{1,4n-1},$$

$$\alpha_{4n+1} = \frac{x_{4n}x_{4n+1}X_{1,4n+2}}{X_{4,4n-1}}, \qquad \alpha_{4n+2} = \frac{x_{4n+2}x_{4n+3}X_{1,4n-1}}{X_{4,4n-1}},$$

$$(n = 1, 2, 3, \cdots), \qquad \alpha_2 = \frac{x_2x_3}{1 + x_2 + x_3}.$$

(3.20) $\begin{Vmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix}$

$$\alpha_{4n-2} = -X_{1,4n-3}, \qquad \alpha_{4n-1} = X_{1,4n-2},$$

$$\alpha_{4n} = \frac{x_{4n-1}x_{4n}X_{1,4n+1}}{X_{4,n-2}}, \qquad \alpha_{4n+1} = \frac{x_{4n+1}x_{4n+2}X_{1,4n-2}}{X_{4,4n-2}},$$

$$(n = 1, 2, 3, \cdots).$$

(3.3)[7] $\begin{Vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{Vmatrix}$

$$\alpha_{2n+2} = \frac{X_{1,2n}X_{1,2n+2}}{x_{2n+2}}, \qquad \alpha_{2n+1} = x_{2n+1},$$

$$(n = 1, 2, 3, \cdots), \qquad \alpha_2 = \frac{1 + x_3}{x_2}.$$

(3.4) $\begin{Vmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix}$

$$\alpha_{4n-2} = \frac{X_{1,4n-3}}{-x_{4n-2}}, \qquad \alpha_{4n-1} = -\frac{1}{X_{1,4n-2}},$$

$$\alpha_{4n} = -\frac{x_{4n-1}x_{4n}X_{1,4n+1}}{X_{1,4n-2}X_{3,4n-1}}, \qquad \alpha_{4n+1} = -\frac{x_{4n+1}}{X_{3,4n-1}},$$

$$(n = 1, 2, 3, \cdots),$$

(3.40) $\begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{Vmatrix}$

$$\alpha_{4n-1} = \frac{X_{1,4n-2}}{-x_{4n-1}}, \qquad \alpha_{4n} = -\frac{1}{X_{1,4n-1}},$$

$$\alpha_{4n+1} = -\frac{x_{4n}x_{4n+1}X_{1,4n+2}}{X_{1,4n-1}X_{3,4n}}, \qquad \alpha_{4n+2} = -\frac{x_{4n+2}}{X_{3,4n}},$$

$$(n = 1, 2, 3, \cdots), \qquad \alpha_2 = -\frac{x_2}{X_{2,1}}.$$

[7] This result was also obtained by Leighton and Wall, loc. cit.

(3.5) $\begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{Vmatrix}$

$$\alpha_{4n-1} = -\frac{1}{X_{1,4n-2}}, \qquad \alpha_{4n} = \frac{X_{1,4n-1}}{x_{4n}},$$

$$\alpha_{4n+1} = -\frac{x_{4n+1}}{X_{3,4n-1}}, \qquad \alpha_{4n+2} = \frac{-x_{4n+2}x_{4n+3}X_{1,4n-1}}{X_{1,4n+2}X_{3,4n-1}},$$

$$(n = 1, 2, 3, \cdots), \qquad \alpha_2 = -\frac{x_2x_3}{X_{1,1}X_{1,2}}.$$

(3.50) $\begin{Vmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix}$

$$\alpha_{4n-2} = -\frac{1}{X_{1,4n-2}}, \qquad \alpha_{4n-1} = \frac{X_{1,4n-2}}{x_{4n-1}},$$

$$\alpha_{4n} = -\frac{x_{4n}}{X_{3,4n-2}}, \qquad \alpha_{4n+1} = -\frac{x_{4n+1}x_{4n+2}X_{1,4n-2}}{X_{1,4n+1}X_{3,4n-2}},$$

$$(n = 1, 2, 3, \cdots).$$

(3.6) $\begin{Vmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{Vmatrix}$

$$\alpha_{4n-1} = \frac{1}{x_{4n-1}}, \qquad \alpha_{4n} = \frac{X_{1,4n-2}X_{1,4n}}{X_{3,4n-2}},$$

$$\alpha_{4n+1} = \frac{x_{4n}x_{4n+1}x_{4n+2}}{X_{3,4n}X_{3,4n-2}}, \qquad \alpha_{4n+2} = -\frac{X_{1,4n}X_{1,4n+2}}{X_{3,4n}},$$

$$(n = 1, 2, 3, \cdots), \qquad \alpha_2 = \frac{X_{1,2}}{X_{2,1}}.$$

(3.7) $\begin{Vmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{Vmatrix}$

$$\alpha_{4n-1} = -\frac{1}{x_{4n-1}}, \qquad \alpha_{4n} = \frac{X_{1,4n-2}X_{3,4n-1}}{-x_{4n-1}x_{4n}X_{1,4n+1}},$$

$$\alpha_{4n+1} = \frac{x_{4n+1}}{X_{1,4n-1}X_{1,4n+1}}, \qquad \alpha_{4n+2} = \frac{X_{1,4n+2}X_{3,4n-1}}{-x_{4n+2}x_{4n+3}X_{1,4n-1}},$$

$$(n = 1, 2, 3, \cdots), \qquad \alpha_2 = \frac{X_{1,1}X_{1,2}}{-x_2x_3}.$$

(3.8) $\begin{Vmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{Vmatrix}$

$$\alpha_{4n-1} = -\frac{1}{X_{1,4n-2}}, \qquad \alpha_{4n} = -\frac{X_{3,4n-2}}{X_{1,4n-2}X_{1,4n}},$$

$$\alpha_{4n+1} = \frac{x_{4n}x_{4n+1}x_{4n+2}}{X_{1,4n}X_{5,4n-2}}, \qquad \alpha_{4n+2} = -\frac{x_{4n+3}X_{3,4n-2}}{X_{5,4n-2}},$$

$$(n = 1, 2, 3, \cdots), \qquad \alpha_2 = -\frac{x_3}{X_{2,1}},$$

(3.9) $\begin{Vmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{Vmatrix}$

$$\alpha_{4n-1} = -\frac{1}{X_{1,4n-2}}, \qquad \alpha_{4n} = -\frac{x_{4n-1}X_{3,4n}}{X_{5,4n-2}},$$

$$\alpha_{4n+1} = -\frac{x_{4n}x_{4n+1}x_{4n+2}}{X_{1,4n}X_{5,4n-2}}, \qquad \alpha_{4n+2} = -\frac{X_{3,4n}}{X_{1,4n}X_{1,4n+2}},$$

$$(n = 1, 2, 3, \cdots), \qquad \alpha_2 = -\frac{X_{2,1}}{X_{1,2}}.$$

(3.11) $$\begin{Vmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{Vmatrix}$$

$$\alpha_{4n-1} = \frac{X_{1,4n-2}}{-x_{4n-2}}, \qquad \alpha_{4n} = \frac{X_{4,4n-2}}{x_{4n-1}x_{4n}X_{1,4n+1}},$$

$$\alpha_{4n+1} = -\frac{x_{4n}x_{4n+1}X_{1,4n-2}}{X_{1,4n+1}X_{3,4n-2}}, \qquad \alpha_{4n+2} = \frac{X_{4,4n-2}}{x_{4n+3}X_{3,4n-2}},$$

$$(n = 1, 2, 3, \cdots), \qquad \alpha_2 = \frac{X_{1,1}}{x_3}.$$

(3.12) $$\begin{Vmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{Vmatrix}$$

$$\alpha_{4n-1} = -\frac{X_{1,4n-2}}{x_{4n-1}}, \qquad \alpha_{4n} = \frac{X_{4,4n-1}}{x_{4n-1}X_{3,4n}},$$

$$\alpha_{4n+1} = -\frac{x_{4n}x_{4n+1}X_{1,4n+2}}{X_{1,4n-1}X_{3,4n}}, \qquad \alpha_{4n+2} = \frac{X_{4,4n-1}}{x_{4n+2}x_{4n+3}X_{1,4n-1}},$$

$$(n = 1, 2, 3, \cdots), \qquad \alpha_2 = \frac{X_{2,1}}{x_2x_3}.$$

(3.13) $$\begin{Vmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{Vmatrix}$$

$$\alpha_{4n-1} = -X_{1,4n-2}, \qquad \alpha_{4n} = \frac{x_{4n-1}X_{3,4n}}{X_{4,4n-1}},$$

$$\alpha_{4n+1} = \frac{x_{4n}x_{4n+1}X_{1,4n+2}}{X_{4,4n-1}}, \qquad \alpha_{4n+2} = -\frac{X_{3,4n}}{x_{4n+2}},$$

$$(n = 1, 2, 3, \cdots), \qquad \alpha_2 = -\frac{X_{2,1}}{x_2}.$$

(3.14) $$\begin{Vmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{Vmatrix}$$

$$\alpha_{4n-1} = -X_{1,4n-2}, \qquad \alpha_{4n} = -\frac{X_{3,4n-2}}{x_{4n}},$$

$$\alpha_{4n+1} = \frac{x_{4n+1}x_{4n+2}X_{1,4n-2}}{X_{4,4n-2}}, \qquad \alpha_{4n+2} = \frac{x_{4n+3}X_{3,4n-2}}{X_{4,4n-2}},$$

$$(n = 1, 2, 3, \cdots), \qquad \alpha_2 = \frac{x_3}{X_{1,1}},$$

(3.15) $$\begin{Vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{Vmatrix}$$

$$\alpha_{4n-1} = x_{4n-1}, \qquad \alpha_{4n} = -\frac{X_{1,4n-2}X_{3,4n-1}}{X_{4,4n-2}},$$

$$\alpha_{4n+1} = \frac{x_{4n-1}x_{4n}x_{4n+1}x_{4n+2}x_{4n+3}}{X_{4,4n-2}X_{4,4n-1}},$$

$$\alpha_{4n+2} = -\frac{X_{1,4n+2}X_{3,4n-1}}{X_{4,4n-1}},$$

$$(n = 1, 2, 3, \cdots), \qquad \alpha_2 = -\frac{X_{1,1}X_{1,2}}{X_{2,1}}.$$

$$(3.16)\qquad \begin{Vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{Vmatrix} \quad \alpha_{4n-1} = -\frac{x_{4n-1}}{X_{1,4n-2}}, \qquad \alpha_{4n} = -\frac{X_{4,4n-1}}{X_{5,4n-2}},$$

$$\alpha_{4n+1} = -\frac{x_{4n-1}x_{4n}x_{4n+1}x_{4n+2}x_{4n+2}}{X_{3,4n-1}X_{5,4n-2}},$$

$$\alpha_{4n+2} = -\frac{X_{4,4n-1}}{X_{1,4n+2}X_{3,4n-1}},$$

$$(n = 1, 2, 3, \cdots), \qquad \alpha_2 = -\frac{X_{2,1}}{X_{1,1}X_{1,2}},$$

$$(3.17)\qquad \begin{Vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{Vmatrix} \quad \alpha_{4n-1} = -\frac{x_{4n-1}}{X_{1,4n-2}}, \qquad \alpha_{4n} = -\frac{X_{4,4n-2}}{X_{1,4n-2}X_{3,4n-1}},$$

$$\alpha_{4n+1} = -\frac{x_{4n-1}x_{4n}x_{4n+1}x_{4n+2}x_{4n+3}}{X_{3,4n-1}X_{5,4n-2}}, \qquad \alpha_{4n+2} = \frac{X_{4,4n-2}}{X_{5,4n-2}},$$

$$(n = 1, 2, 3, \cdots), \qquad \alpha_2 = -\frac{X_{1,1}}{X_{2,1}},$$

$$(3.18)\qquad \begin{Vmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{Vmatrix} \quad \alpha_{4n-1} = \frac{1}{x_{4n-1}}, \qquad \alpha_{4n} = -\frac{X_{5,4n-2}}{x_{4n-1}X_{3,4n}},$$

$$\alpha_{4n+1} = \frac{x_{4n}x_{4n+1}x_{4n+2}}{X_{3,4n}X_{3,4n-2}}, \qquad \alpha_{4n+2} = -\frac{X_{5,4n-2}}{x_{4n+3}X_{3,4n-2}},$$

$$(n = 1, 2, 3, \cdots), \qquad \alpha_2 = -\frac{X_{2,1}}{x_3},$$

$$(3.19)\qquad \begin{Vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{Vmatrix} \quad \alpha_{4n-1} = x_{4n-1}, \qquad \alpha_{4n} = -\frac{X_{5,4n-2}}{X_{4,4n-1}},$$

$$\alpha_{4n+1} = \frac{x_{4n-1}x_{4n}x_{4n+1}x_{4n+2}x_{4n+3}}{X_{4,4n-1}X_{4,4n-2}}, \qquad \alpha_{4n+2} = -\frac{X_{5,4n-2}}{X_{4,4n-2}},$$

$$(n = 1, 2, 3, \cdots), \qquad \alpha_2 = -\frac{X_{2,1}}{X_{1,1}}.$$

Some of the preceding convergence criteria have particularly simple corollaries. The number of each corollary below corresponds to that of the transformation from which it results.

COROLLARY 3.1. *The continued fraction* (1.1) *converges if*

$$|x_{4n-2}| < \tfrac{1}{4}, \qquad |x_{2n+1}| \leqq \tfrac{1}{5}, \qquad |x_{4n}| \geqq 4, \qquad (n = 1, 2, 3, \cdots).$$

COROLLARY 3.10. *The continued fraction* (1.1) *converges if*

$$|x_{2n}| \leqq \tfrac{1}{5}, \qquad |x_{4n+1}| < \tfrac{1}{4}, \qquad |x_{4n-1}| \geqq 4, \qquad (n = 1, 2, 3, \cdots).$$

COROLLARY 3.2. *The continued fraction* (1.1) *converges if*

$$m \leqq |1 + x_{4n-1}| \leqq \tfrac{1}{4}, \qquad m \leqq |1 + x_{4n}| \leqq \tfrac{1}{4}, \qquad (m > 0),$$

$$|x_{4n+1}| \leqq \frac{m}{1 + |4x_{4n}|}, \qquad |x_{4n-2}| \leqq \frac{m}{1 + |4x_{4n-1}|}, \qquad (n = 1, 2, 3, \cdots).$$

COROLLARY 3.20. *The continued fraction* (1.1) *converges if*

$$m \leqq |1 + x_{4n-2}| \leqq \tfrac{1}{4}, \qquad m \leqq |1 + x_{4n-1}| \leqq \tfrac{1}{4}, \qquad (m > 0),$$

$$|x_{4n}| \leqq \frac{m}{1 + |4x_{4n-1}|}, \qquad |x_{4n+1}| \leqq \frac{m}{1 + |4x_{4n+2}|}, \qquad (n = 1, 2, 3, \cdots).$$

COROLLARY 3.3.[8] *The continued fraction* (1.1) *converges if*

$$|x_{2n+2}| \leqq \tfrac{1}{4}, \qquad |x_{2n+2}| \geqq \tfrac{25}{4}, \qquad (n = 1, 2, 3, \cdots), \qquad |x_2| \geqq 5.$$

COROLLARY 3.4. *The continued fraction* (1.1) *converges if*

$$0 < m \leqq \left|\frac{1 + x_{4n-2}}{x_{4n-2}}\right| \leqq \tfrac{1}{4}, \qquad |x_{4n-1}| \geqq 5,$$

$$|x_{4n}| \leqq 1 - 2m, \qquad |x_{4n+1}| \leqq 2m/5, \qquad (n = 1, 2, 3, \cdots).$$

COROLLARY 3.40. *The continued fraction* (1.1) *converges if*

$$0 < m \leqq \left|\frac{1 + x_{4n-1}}{x_{4n-1}}\right| \leqq \tfrac{1}{4}, \qquad |x_{4n}| \geqq 5, \qquad |x_{4n+1}| \leqq 1 - 2m,$$

$$|x_{4n+2}| \leqq 2m/5, \qquad (n = 1, 2, 3, \cdots), \qquad |x_2| \leqq \tfrac{1}{15}.$$

COROLLARY 3.5. *The continued fraction* (1.1) *converges if*

$$|x_{4n-1}| \geqq 5, \qquad 0 < m \leqq \left|\frac{1 + x_{4n}}{x_{4n}}\right| \leqq \tfrac{1}{4}, \qquad |x_{4n+1}| \leqq 2m/5,$$

$$|x_{4n+2}| \leqq 1 - 2m, \qquad (n = 1, 2, 3, \cdots), \qquad |x_2| \leqq \tfrac{1}{3}.$$

COROLLARY 3.50. *The continued fraction* (1.1) *converges if*

$$|x_{4n-2}| \geqq 5, \qquad 0 < m \leqq \left|\frac{1 + x_{4n-1}}{x_{4n-1}}\right| \leqq \tfrac{1}{4},$$

$$|x_{4n}| \leqq 2m/5, \qquad |x_{4n+1}| \leqq 1 - 2m, \qquad (n = 1, 2, 3, \cdots).$$

COROLLARY 3.6. *The continued fraction* (1.1) *converges if*

$$M - 1 \geqq |x_{4n-1}| \geqq 4, \quad |x_{2n}| \geqq 29M/5, \quad |x_{4n+1}| \leqq 4/25, \quad (n = 1, 2, 3, \cdots).$$

COROLLARY 3.7. *The continued fraction* (1.1) *converges if*

$$|x_{4n-1}| \geqq 4, \qquad 0 < m \leqq \left|\frac{1 + x_{2n+2}}{x_{2n+2}}\right| \leqq \tfrac{1}{10},$$

---

[8] This result was also obtained by Leighton and Wall, loc. cit.

$$|x_{4n+1}| \leqq m^2/4, \qquad (n = 1, 2, 3, \cdots), \qquad m \leqq \left|\frac{1+x_2}{x_2}\right| \leqq \tfrac{1}{3}.$$

The analogues of Theorem 2.2 resulting from Theorem 3.1 and the corollaries above are evident.

4. Results of permuting convergents three at a time

When we permute convergents three at a time we have the following theorem. Here as in the preceding section the matrices indicate the particular transformation that yields the given results.

THEOREM 4.1. *The continued fraction* (1.1) *converges if*

$$|\alpha_n| \leqq \tfrac{1}{4} \qquad (n = 2, 3, 4, \cdots)$$

where the value of α_n *is given by any one of the following sets of relations.*

(4.1) $\begin{Vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{Vmatrix}$ $\alpha_{3n} = \dfrac{1}{x_{3n}}, \quad \alpha_{3n+1} = -\dfrac{x_{3n+1}}{X_{2,3n}},$

$\alpha_{3n+2} = -\dfrac{x_{3n+2}}{X_{2,3n}}, \quad (n = 1, 2, 3, \cdots), \quad \alpha_2 = -\dfrac{x_2}{X_{1,1}},$

(4.10) $\begin{Vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{Vmatrix}$ $\alpha_{3n-1} = \dfrac{1}{x_{3n-1}}, \quad \alpha_{3n} = -\dfrac{x_{3n}}{X_{2,3n-1}},$

$\alpha_{3n+1} = -\dfrac{x_{3n+1}}{X_{2,3n-1}}, \qquad (n = 1, 2, 3, \cdots).$

(4.2) $\begin{Vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{Vmatrix}$ $\alpha_{3n} = -\dfrac{X_{2,3n-1}}{x_{3n}}, \quad \alpha_{3n+1} = \dfrac{x_{3n+1}x_{3n+2}}{X_{3,3n-1}},$

$\alpha_{3n+2} = -\dfrac{X_{2,3n-1}}{X_{3,3n-1}}, \quad (n = 1, 2, 3, \cdots),$

$\alpha_2 = -\dfrac{1}{1+x_2}.$

(4.20) $\begin{Vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{Vmatrix}$ $\alpha_{3n} = -\dfrac{X_{2,3n}}{X_{3,3n-1}}, \quad \alpha_{3n+1} = \dfrac{x_{3n}x_{3n+1}}{X_{3,3n-1}},$

$\alpha_{3n+2} = -\dfrac{X_{2,3n}}{x_{3n+2}}, \quad (n = 1, 2, 3, \cdots), \quad \alpha_2 = \dfrac{1+x_2}{-x_2}.$

(4.3) $\begin{Vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{Vmatrix}$ $\alpha_{3n} = -\dfrac{X_{3,3n-1}}{X_{2,3n}}, \quad \alpha_{3n+1} = \dfrac{x_{3n}x_{3n+1}x_{3n+2}}{X_{2,3n-1}X_{2,3n}},$

$\alpha_{3n+2} = -\dfrac{X_{3,3n-1}}{X_{2,3n-1}}, \quad (n = 1, 2, 3, \cdots),$

$\alpha_2 = -\dfrac{1}{1+x_2}.$

As immediate results of the first four of the above conditions we have respectively the following corollaries.

COROLLARY 4.1.[9] *The continued fraction* (1.1) *converges if*

$$|x_2| \leqq \tfrac{1}{5}, \quad |x_{3n}| \geqq 4, \quad |x_{3n+1}| \leqq \tfrac{1}{6}, \quad |x_{3n+2}| \leqq \tfrac{1}{6}, \qquad (n = 1, 2, 3, \cdots).$$

COROLLARY 4.10. *The continued fraction* (1.1) *converges if*

$$|x_{3n}| \leqq \tfrac{1}{6}, \quad |x_{3n+1}| \leqq \tfrac{1}{6}, \quad |x_{3n+2}| \geqq 4, \qquad (n = 1, 2, 3, \cdots).$$

COROLLARY 4.2. *The continued fraction* (1.1) *converges if*

$$0 < \left|\frac{1 + x_{3n}}{x_{3n}}\right| \leqq \tfrac{1}{5}, \quad |x_{3n+1}| \leqq |x_{3n}|/20,$$

$$|x_{3n+2}| \geqq \tfrac{5}{4}, \qquad (n = 1, 2, 3, \cdots), \qquad |x_2| \geqq 5.$$

COROLLARY 4.20. *The continued fraction* (1.1) *converges if*

$$|x_{3n}| \geqq \tfrac{5}{4}, \quad |x_{3n+1}| \leqq |x_{3n+2}|/20,$$

$$0 < \left|\frac{1 + x_{3n+2}}{x_{3n+2}}\right| \leqq \tfrac{1}{5}, \qquad (n = 1, 2, 3, \cdots), \qquad 0 < \left|\frac{1 + x_2}{-x_2}\right| \leqq \tfrac{1}{4}.$$

The analogues of Theorem 2.2 are again, as in section 3, immediately evident.

In conclusion we again stress the fact that each of the formulae (3.1) to (3.19) and (4.1) to (4.3) inclusive, not only leads to new convergence criteria, but also defines a transformation of the continued fraction (1.1) into an equivalent continued fraction.

HARVARD UNIVERSITY,
THE RICE INSTITUTE.

[9] The same result was obtained by a different method by Leighton and Wall, loc. cit.

ANNALS OF MATHEMATICS
Vol. 39, No. 4, October, 1938

A THEORY OF CONNECTIVITY IN TERMS OF GRATINGS[1]

BY J. W. ALEXANDER

(Received April 28, 1938)

1. Our original intention was to develop a simplified theory of connectivity applicable to any locally bicompact, topological space. However, it was found that the essentially combinatory character of the theory could be brought out more sharply if we looked at the problem from a more general point of view and thought of a space not as a collection of points but as a primitive entity appearing as an element in a boolean algebra.

In §3 we shall give a very general definition of a space, without introducing the notion of a point, and shall say what it means for a space to be normal, bicompact, or even locally bicompact. The points of a space will often be hiding in the background, waiting to be constructed by suitable limiting processes. Whether they are actually there or not will, however, be no concern of ours, as the entities with which we shall be dealing—chains, cycles, etc.—will all be obtainable by finite processes, without a passage to the limit.

On a space x, we shall construct certain configurations, called gratings, §4. A grating will be analogous to a complex (or, more precisely, to a family of complexes) determining a manifold. It will have both an homology theory and a theory of intersecting chains. Hence we shall be able to define its connectivity groups and to imbed these last in a connectivity ring. By constructing suitable invariant gratings on the space x we shall arrive at invariant systems of chains and cycles leading to invariant connectivity groups and rings. In particular, we shall discuss the theories of the open, the closed, and the bicompact chains. If a space is normal, the first two theories will lead to the same connectivity ring, §20, if bicompact, the last two will, §22. The theory of the bicompact chains will be the one giving the most satisfactory generalization of the classical theory.

Given a normal, bicompact space z, subject to one very natural restriction, and given, further, a closed subspace x of z and the open complement y of x, we shall obtain relations between the chain systems of the three spaces x, y, and z, §23. We shall also obtain a relation of duality between the invariants of the complementary spaces x and y when the containing space z is "simply connected."

We shall pay more attention to questions of formal manipulation than would have been necessary if we had merely wanted to *define* the invariants of a space.

[1] This paper is the outgrowth of a short series of lectures delivered by the author at Cambridge University in April 1936. It includes, in greatly revised form, the material outlined in two abstracts *On the Connectivity Ring of a Bicompact Space*, Proc. of Nat. Acad. of Sci., vol. 22 (1936), pp. 300–303 and pp. 381–384.

In fact, one of our main objectives will be to develop an efficient working technique adapted to the solution of concrete topological problems. Thus, in particular, we shall include a number of theorems which will be helpful in evaluating the invariants of certain special classes of spaces.

2. We remind the reader that a *boolean algebra* A consists of a set of elements x and of two binary operators $\smile$ and $\frown$, called *union* and *intersection* respectively, such that:

1) The union $x_1 \smile x_2$ and intersection $x_1 \frown x_2$ of two elements of A are elements of A.

2) Each of the operations $\smile$ and $\frown$ is associative and commutative, and each is distributive with respect to itself and with respect to the other.

3) There exist two elements 0 and 1 satisfying the relations

$$0 \smile x = x, \qquad 0 \frown x = 0, \qquad 1 \smile x = 1, \qquad 1 \frown x = x.$$

4) To each element x there corresponds an element y, called the *complement* of x, such that we have

$$x \smile y = 1, \qquad x \frown y = 0.^{2}$$

If the elements of a boolean algebra are sets of objects and if the symbols $\smile$ and $\frown$ denote logical addition and multiplication respectively, it will be possible, by extension, to define the union and intersection of an infinite set of elements. However, we shall not be able to do this in the general case unless we also enlarge the algebra.

It will be convenient to introduce an *intersection operator* $(^z)$ corresponding to each element z of A. The operator $(^z)$ will transform an element x of A into its intersection with z, $x^z = x \frown z$. Owing to the distributive properties of union and intersection we shall have

$$(x \smile y)^z = x^z \smile y^z, \qquad (x \frown y)^z = x^z \frown y^z,$$
$$x^{y \smile z} = x^y \smile x^z, \qquad x^{y \frown z} = x^y \frown x^z.^{3}$$

3. We shall start with an arbitrary boolean algebra A and an arbitrary preferred class of elements p of A such that:

1) The elements 0 and 1 belong to the class;

2) The union and intersection of two elements of the class belong to the class.

The complements of the elements p will be called the elements q. They will also form a preferred class with the properties 1) and 2).

---

[2] The four postulates, as stated above, are not independent. Various sets of independent postulates for boolean algebras have been given by E. V. Huntington: *Sets of independent postulates for the algebra of logic.* Transactions of Amer. Math. Soc., vol. 5 (1904), pp. 288-309.

[3] Of course, we might also introduce an analogous *union operator*, but this will not be necessary.

A *space* will simply be any element x of the algebra A. A space x will *meet* a space y if we have $x \frown y \neq 0$; it will *dominate* y, $x \supseteq y$, if we have either of the following equivalent relations:

$$x \smile y = x, \qquad x \frown y = y,$$

in which case we shall also say that y is a *subspace* of x. The *closed subspaces* p^x [*open subspaces* q^x] of x will be the intersections with x of the preferred elements p [of the preferred elements q],

$$p^x = x \frown p, \qquad q^x = x \frown q.$$

LEMMA [3:1]. *If y is a closed [open] subspace of x and z a closed [open] subspace of y then z is a closed [open] subspace of x.*

For

$$y = p_1 \frown x, z = p_2 \frown y \quad \rightarrow \quad z = (p_1 \frown p_2) \frown x,$$

and similarly for the other case.

LEMMA [3:2]. *The union and intersection of two closed [open] subspaces x and y of a space z are both closed [open].*

For

$$x = p_1 \frown z, y = p_2 \frown z \quad \rightarrow$$

$$x \smile y = (p_1 \smile p_2) \frown z, \qquad x \frown y = (p_1 \frown p_2) \frown z.$$

Every space x will be the unit element of a sub-algebra X of A consisting of all subspaces of x. We shall say that two spaces x and y are *homeomorphic* provided there exists a simple isomorphism between their algebras X and Y such that the closed subspaces of x are paired in the isomorphism with the closed subspaces of y (and, therefore, also the open subspaces of x with the open subspaces of y).

There is one special case in which we shall want to define the intersection of an infinite set of spaces. We shall say that the intersection of a finite *or infinite* set of spaces x_s is 0 provided 0 is the only space dominated by all the spaces x_s of the set. If the spaces x_s are finite in number, this definition will obviously be consistent with the direct definition, $x_1 \frown x_2 \frown \cdots \frown x_n = 0$, in terms of the intersection operator. A space x will be said to be *bicompact* if to every set of closed subspaces p_s^x of x such that their intersection is 0 there corresponds a finite subset having this same property,

$$p_{s_1}^x \frown p_{s_2}^x \frown \cdots \frown p_{s_n}^x = 0.^4$$

A space will be said to be *locally bicompact* if it is homeomorphic with an open subspace of a bicompact space.[5]

[4] Of course, this is merely the Heine-Borel-Lebesgue condition restated in terms of closed sets.

[5] Some such designation as *semi-bicompact* might, perhaps, be more appropriate. However, for point spaces of the more usual types, the above definition is equivalent to the classical one, in spite of its different formal appearance.

THEOREM [3:3]. *Every bicompact space is locally bicompact.*

For a space x is an open subspace of itself, $x \frown 1 = x$.

THEOREM [3:4]. *Every closed subspace y of a bicompact space x is bicompact.*

For suppose there were a set of closed subspaces p_s^y of y such that their intersection was 0 and such that they contained no finite subset with the intersection 0. Then these same subspaces, regarded as closed subspaces of x, [3:1], would imply that x was not bicompact, contrary to hypothesis.

COROLLARY [3:5]. *Every open subspace y of a locally bicompact space x is locally bicompact.*

For x is of the form $x = q_1 \frown z$, where z is bicompact, and y of the form $y = q_2 \frown x$. Therefore we have $y = (q_1 \frown q_2) \frown z$.

THEOREM [3:6]. *The union $z = x \smile y$ of two bicompact spaces x and y is bicompact.*

PROOF. We must show that to every set of closed subspaces $p_s^z = p_s \frown z$ of z such that their intersection is 0 there corresponds a finite subset σ having this same property. Now, corresponding to the subspaces p_s^z there will be certain closed subspaces $p_s^x = p_s \frown x$ of x such that their intersection is surely also 0. Therefore, since x is bicompact, there will be a finite subset σ^x of the subspaces p_s^x satisfying a relation of the form

$$p_{s_1}^x \frown p_{s_2}^x \frown \cdots \frown p_{s_m}^x = 0. \tag{3:1}$$

Moreover, by a similar argument, there will be a finite subset σ^y of the closed subspaces $p_s^y = p_s \frown y$ of y satisfying a relation of the form

$$p_{s_{m+1}}^y \frown p_{s_{m+2}}^y \frown \cdots \frown p_{s_{m+n}}^y = 0. \tag{3:2}$$

The subset composed of the spaces $p_{s_i}^z$ corresponding to the spaces $p_{s_i}^x$ of σ^x together with the spaces $p_{s_{m+1}}^z$ corresponding to the spaces p_s^y of σ^y will be the desired subset σ. For if σ had an intersection u different from 0, we would have either $u \frown x \neq 0$ or $u \frown y \neq 0$ (because $u = (u \frown x) \smile (u \frown y)$). Moreover, the spaces of the set σ^x would all dominate $u \frown x$ and the spaces of the set σ^y would all dominate $u \frown y$. Therefore we would arrive at a contradiction with either (3:1) or (3:2).

A space x will be said to be *normal* provided that to each pair of closed subspaces p_1^x and p_2^x with the intersection 0 there correspond two open subspaces q_1^x and q_2^x with the intersection 0 such that q_1^x and q_2^x dominate p_1^x and p_2^x respectively.[6]

THEOREM [3:7]. *Every closed subspace of a normal space is normal.*

By [3:1] and the definition of normality.

If the underlying boolean algebra A consists of all the subsets of a point set $[P]$ and if the operations $\smile$ and $\frown$ denote logical addition and multiplication respectively, we shall say that the elements x of the algebra are *point spaces*.

[6] Cf. the classical definition of normality for ordinary topological point spaces.

4. A *covering pair* (a, c) of a space x will be any ordered pair of subspaces a and c of x such that their union is x, $a \smile c = x$. The *barrier* b of a covering pair (a, c) will be the intersection of the two members of the pair, $a \frown c = b$. The *positive* and *negative sides* of the barrier b will be the subspaces c and a respectively. An *indexed pair* (a_s, c_s) will be a covering pair in association with an index s. If two indexed pairs (a_s, c_s) and (a_t, c_t) differ in index only we shall say that each is a *replica* of the other.

A *grating* Γ of x will be an arbitrary set of indexed pairs (a_s, c_s) of x without repeated indices. A grating may be finite or infinite. If two or more of its indexed pairs are replicas of one another, we shall call the grating a *singular grating*, or a grating with *repeated* covering pairs.

If Γ_1 and Γ_2 are any two gratings of x such that every indexed pair of Γ_1 belongs to Γ_2 we shall say that Γ_1 is an *approximation* of Γ_2 and that Γ_2 is a *refinement* of Γ_1. The *union* $\Gamma = \Gamma_1 \smile \Gamma_2$ of two different gratings Γ_1 and Γ_2 of x will be the common refinement of Γ_1 and Γ_2 consisting of the combined indexed pairs of the two gratings.

A typical example of a grating will be the *coördinate grating* of the point space R^n of n real variables ξ_i. This grating will be determined by the n coordinate hyperplanes $\xi_i = 0$ and will consist of n indexed pairs (a_s, c_s), such that a_s, b_s, and c_s are the subspaces determined by the conditions $\xi_s \leqq 0$, $\xi_s = 0$, and $\xi_s \geqq 0$ respectively.

5. With every grating Γ we are going to associate an abstract ring Π, called the *ring of chains* of Γ. As a preliminary step, however, we shall first set up an auxiliary ring Π^a formed by taking the ring of all the integers and by adjoining as new and independent marks the symbols a_s, b_s, and c_s associated with the various indexed pairs of Γ. We shall assume that the adjoined marks commute with the integers but not with one another, so that a general element of the auxiliary ring will be representable by a polynomial K in the adjoined marks, with integer coefficients. The order of the factors in the terms of K will, of course, be significant, since the adjoined marks are not commutative. Each polynomial K will represent a distinct element of Π^a, and the sum, difference, and product of two elements of Π^a will be formed according to the usual polynomial algorithms.

6. For the moment, we shall ignore the underlying grating Γ and examine the auxiliary ring Π^a from a purely formal point of view. Every polynomial K of Π^a will be a linear combination of monomials of the form

$$F = z_{s_1} z_{s_2} \cdots z_{s_n} \qquad (z_{s_i} = a_{s_i},\ b_{s_i} \text{ or } c_{s_i}), \tag{6:1}$$

where, by convention, we shall want to include the monomial $F = 1$, corresponding to the case $n = 0$. Moreover, the monomials (6:1) will be linearly independent. Therefore, a linear operator operating on the polynomials K

will be fully determined when we state how it operates on the monomials (6:1) alone.

A polynomial [monomial] will be called a *τ-polynomial* [*τ-monomial*], or a polynomial [monomial] of *type* τ if it is homogeneous of degree τ in the combined marks b_s (as distinguished from the marks a_s and c_s). Thus, a general polynomial K will admit a unique decomposition into a sum of the form

$$K = \sum_{\tau} K^{\tau} \qquad (\tau = 0, 1, 2, \cdots), \tag{6:2}$$

when K is a τ-polynomial. We shall call the polynomials K^{τ} the *τ-components* of K. For any particular choice of K, at most a finite number of τ-components will be non-vanishing. (It may be well to observe that the polynomial 0 should be thought of as of *all* types simultaneously.)

It will be convenient to think of $(^{\tau})$ as a *type operator* transforming each polynomial K into its τ-component K^{τ}. Moreover, it will be convenient to introduce a *conjugation operator* $(^*)$ transforming each polynomial K with components K into the *conjugate*

$$K^* = \sum_{\tau} (-1)^{\tau} K^{\tau}, \tag{6:3}$$

of K, formed by merely changing the signs of all the components of K of odd types. The type operators and the conjugation operator will obviously be linear.

$$(K \pm L)^{\tau} = K^{\tau} \pm L^{\tau} \qquad (K \pm L)^* = K^* \pm L^*, \tag{6:4}$$

and we shall have the following simple relations,

$$(KL)^* = K^* L^* \tag{6:5}$$

and

$$K^{**} = K \qquad [K^{**} = (K^*)^*]. \tag{6:6}$$

7. A more significant operator will be the *boundary operator* $(')$ which will transform a polynomial K of Π^a into a polynomial K', called the *boundary* of K. We shall assume that the boundary operator is linear:

$$(K \pm L)' = K' \pm L', \tag{7:1}$$

so that it will be sufficient for us to define the boundaries of the monomials (6:1). The cases $F = 1, a_s, b_s, c_s$ will be disposed of by writing

$$1' = 0, \qquad a_s' = b_s, \qquad b_s' = 0, \qquad c_s' = -b_s. \tag{7:2}$$

The remaining cases, where F contains two or more factors, will be treated by induction with respect to the number of factors. We shall write F in the form zL, so as to exhibit its first factor z ($z = a_s, b_s$, or c_s), and shall then define F' by the recursion formula

$$(zL)' = z'L + z^*L'. \tag{7:3}$$

LEMMA [7:1]. *Relation* (7:3) *will continue to be valid if we allow the symbol* L *to represent any polynomial whatever of the ring* Π^a

For L can represent any monomial (6:1) with the possible exception of 1, since $F = zL$ is any monomial with at least two factors. Moreover, L can also represent the monomial 1, as may be verified by inspection. Therefore, finally, L can represent any polynomial whatever, since the boundary operator is linear.

THEOREM [7:2]. *If* K *and* L *are arbitrary polynomials of the ring* Π^a *the boundary of their product is*

$$(KL)' = K'L + K^*L'. \tag{7:4}$$

PROOF. Since the boundary and conjugation operators are both linear, the problem reduces to the case where K is a monomial. The sub-case $K = 1$ is trivial. The remaining sub-cases will be treated by induction with respect to the number of factors in the monomial K. This time we shall write $K = Fz$, so as to exhibit the last factor z of K. Then, by the hypothesis of the induction, we shall have

$$(KL)' = (F \cdot zL)' = F' \cdot zL + F^*(zL)' = F'zL + F^*z'L + F^*z^*L' = K'L + K^*L'.$$

By repeated applications of (7:3) the following formula for the boundary of any monomial $F = z_{s_1}z_{s_2} \cdots z_{s_n}$ other than 1 may easily be obtained:

$$F' = (z_{s_1}z_{s_2} \cdots z_{s_n})' = \sum_i (-1)^{\tau(z_{s_1}z_{s_2}\cdots z_{s_{i-1}})} z_{s_1} \cdots z_{s_{i-1}} z'_{s_i} z_{s_{i+1}} \cdots z_{s_n}. \tag{7:5}$$

Here we must, of course, make the convention that the type $\tau(z_{s_1} \cdots z_{s_{i-1}})$ of the product of the first $i - 1$ factors is 0 when $i - 1$ vanishes. By (7:5) the boundary of a τ-monomial is a linear combination of $(\tau + 1)$-monomials.

THEOREM [7:3]. *The boundary of a* τ*-polynomial is a* $(\tau + 1)$*-polynomial.* (By (7:1) and (7:5).)

THEOREM [7:4]. *The boundary of the conjugate of a polynomial* K *is the negative of the conjugate of the boundary of* K,

$$(K^*)' = -(K')^*. \tag{7:6}$$

Since we have

$$(K^*)' = \left\{\sum_\tau (-1)^\tau K^\tau\right\}' = -\left\{\sum_\tau (-1)^{\tau+1}(K^\tau)'\right\} = -(K')^*.$$

THEOREM [7:5]. *The boundary of the boundary of a polynomial* K *always vanishes,*

$$K'' = 0. \tag{7:7}$$

The proof reduces, as usual, to the case where K is a monomial. The subcase $K = 1$ is again trivial. In the other subcases we write $K = zL$ and proceed by induction. We have

$$K'' = (zL)'' = (z'L + z^*L')' = z''L + (z')^*L' + (z^*)'L' + z^*L''.$$

Moreover, the first term in the last member vanishes, by (7:2), the second and third terms cancel, by (7.6), and the last term vanishes, by the hypothesis of the induction. Therefore, finally, K'' itself must vanish.

8. A set of polynomials K_α of Π^a *generates* a sub-ring Π_α^a of Π^a consisting of all linear combinations (with integer coefficients) of expressions of the form

$$K_{\alpha_1}K_{\alpha_2}\cdots K_{\alpha_n} \qquad (n \text{ arbitrary}), \tag{8:1}$$

where the K_{α_1}'s are arbitrary polynomials of the set. It also *determines* a (two-sided) ideal Ω_α^a consisting of all linear combinations (with integer coefficients) of expressions of the form

$$LK_\alpha M, \tag{8:2}$$

where K_α is in the set and where L and M are arbitrary polynomials of Π^a. We shall use the conventional notation $K \equiv 0$ (or, more explicitly, $K \equiv 0$, mod Ω_α^a) to indicate that the polynomial K belongs to the ideal Ω_α^a. The notation $K \equiv L$ will, of course, indicate that K and L belong to the same residue class, modulo Ω_α^a. The congruences $K_\alpha \equiv 0$ formed by identifying the determining polynomials K_α of Ω_α^a with 0 will be said to form a *determining set* of congruences of Ω_α^a.

A sub-ring [ideal] will be said to be *preferred* if the components and boundary (hence, also, the conjugate) of every polynomial of the sub-ring [ideal] belong to the sub-ring [ideal].

THEOREM [8:1]. *A necessary and sufficient condition that a sub-ring* Π_α^a [*an ideal* Ω_α^a] *generated by* [*determined by*] *a set of polynomials* K_α *be preferred is that the components* K_α^τ *and boundary* K_α' *of every polynomial of the set belong to the sub-ring* [*ideal*].

PROOF. The necessity of the condition follows from the very definition of a preferred sub-ring [ideal]; its sufficiency can be proved as follows. Every polynomial K of Π_α^a is a linear combination of expressions of the form (8:1). Therefore, the τ-components K^τ and boundary K' of K are linear combinations of expressions of the form

$$K_{\alpha_1}^{\tau_1}K_{\alpha_1}^{\tau_1}\cdots K_{\alpha_n}^{\tau_n} \qquad (\tau_1+\tau_2+\cdots+\tau_n=\tau),$$

and of the form

$$\begin{aligned}(K_{\alpha_1}K_{\alpha_2}\cdots K_{\alpha_n}) = {} & K'_{\alpha_1}K_{\alpha_2}K_{\alpha_3}\cdots K_{\alpha_n} \\ & + K^*_{\alpha_1}K'_{\alpha_2}K_{\alpha_3}\cdots K_{\alpha_n} \\ & \cdots\cdots\cdots\cdots\cdots \\ & + K^*_{\alpha_1}K^*_{\alpha_2}K^*_{\alpha_3}\cdots K'_{\alpha_n}\end{aligned}$$

respectively. The components and boundary of K therefore surely belong to Π_α^a since the components, boundaries (and, therefore, also the conjugates) of the polynomials K_α belong to Π_α^a. Similarly, every polynomial K of the ideal Ω_α^a is a linear combination of expressions of the form (8:2). Therefore, the

τ-components K^τ and boundary K' of K are linear combinations of expressions of the form

$$L^\mu K^\sigma_\alpha M^\nu \qquad (\mu + \sigma + \nu = \tau),$$

and of the form

$$(LK_\alpha M)' = L'K_\alpha M + L^* K'_\alpha M + L^* K^*_\alpha M'$$

respectively. Therefore, the components and boundary of K belong to Ω^a_α.

Let Π^a_α be any preferred sub-ring of Π^a and Ω^a_β any preferred ideal. Moreover, let $\Pi^a_\alpha \bmod \Omega^a_\beta$ be the ring of the residue classes of Π^a_α, modulo Ω^a_β. Then we shall define the τ-components and boundary of the residue class containing the polynomial K to be the residue classes containing the τ-components K^τ and the residue class containing the boundary K' of K respectively. This definition will evidently be independent of the choice of K within its class, since a congruence $K \equiv L$ will imply the congruences $K^\tau \equiv L^\tau$ and the congruence $K' \equiv L'$.

9. The ring of chains Π of the grating Γ will be formed by reducing the auxiliary ring Π^a modulo a suitable preferred ideal Ω^a. The ideal Ω^a will be suggested by the following heuristic consideration.

We shall fix our attention on the case where Γ is the coordinate grating of the space R^n of n real variables (cf. §4). Then the coördinate hyperplanes $\xi_s = 0$ determining the grating Γ will subdivide the space R^n into a complex Θ such that a cell of the complex will be determined by imposing on each of the coördinates ξ_s one of the three conditions

(9:1) $$\xi_s \leqq 0, \qquad \xi_s = 0, \qquad \xi_s \geqq 0.$$

We shall use the notation

(9:2) $$F = z_1 z_2 \cdots z_n$$

to represent a cell of Θ, where z_s will be either a_s, b_s, or c_s according as we have $\xi_s \leqq 0$, $\xi_s = 0$, or $\xi_s \geqq 0$. If the number of symbols b_s appearing in F is τ, the *dimensionality* of the cell will, of course, be $n - \tau$. By a slight departure from the current terminology, we shall call the cell F a τ-cell rather than an $(n - \tau)$-cell.[7]

Now, the complex Θ has both an homology and an intersection theory, since it is constructed in a space R^n of the manifold type. We shall obtain the ideal Ω^a for the special grating under consideration by expressing the chains of Θ symbolically in terms of the polynomials of Π^a. To each chain of Θ there will correspond a class of polynomial symbols. The class corresponding to the chains 0 of Θ will be the desired ideal Ω^a.

[7] In the present theory it is not the concept of dimensionality so much as the dual concept which is significant.

Let us associate the mark 1 with the entire space R^n and the marks a_s, b_s, c_s with the subspaces $\xi_s \leqq 0$, $\xi_s = 0$, and $\xi_s \geqq 0$ respectively. Then the geometry of chains in the space R^n suggests, at once, that we make the following identifications which we shall first write, then interpret:

$$a_s + c_s \equiv 1, \tag{9:3}$$

$$a_s a_s \equiv a_s, \quad a_s b_s \equiv b_s, \quad b_s a_s \equiv 0, \quad b_s b_s \equiv 0, \tag{9:4}$$

$$a_s a_t \equiv a_t a_s, \quad a_s b_t \equiv b_t a_s, \quad b_s b_t \equiv -b_t b_s, \qquad (s \neq t). \tag{9:5}$$

Let us also write, for future reference, the additional relations

$$a_s c_s \equiv c_s a_s \equiv 0, \quad b_s c_s \equiv b_s, \quad c_s b_s \equiv 0, \quad c_s c_s \equiv c_s, \tag{9:4'}$$

$$a_s c_t \equiv c_t a_s, \quad b_s c_t \equiv c_t b_s, \quad c_s c_t \equiv c_t c_s, \qquad (s \neq t), \tag{9:5'}$$

which are immediately derivable from (9:4) and (9:5) respectively, taken in combination with (9:3).

The geometrical interpretation of Relations (9:3) is immediate. We are merely identifying the 0-chain consisting of the entire space R^n with the refinement of itself consisting of the two half spaces $\xi_s \leqq 0$ and $\xi_s \geqq 0$. To interpret Relations (9:4) and (9:5), let us recall that in the classical theory of intersecting chains the formal product of two chains corresponds, in general, to their geometrical intersection, with suitable conventions about orientation. However, there are certain exceptional cases in which the chains are in singular positions with respect to one another and in which a special convention is needed to define their formal product. Relations (9:5), dealing with marks in different indices s and t, correspond to the general case. The products appearing in their various members symbolize the oriented intersections of the spaces represented by the marks, and the relations themselves tell us how the orientations are affected when we change the order of the marks. Relations (9:4), dealing with marks in the same index s, correspond to the exceptional cases. Putting it simply, they symbolize the convention that we are to give the chain corresponding to the first of the two marks a 'virtual' displacement in the direction of increasing ξ_s before we form its geometrical intersection with the chain corresponding to the other mark. We shall see in just a moment (§11) that the ideal Ω^a determined by (9:3), (9:4), and (9:5) is precisely the one required to make the ring Π^a mod Ω^a simply isomorphic with the ring of chains of Θ.

The rough argument outlined above suggests the rigorous formal procedure which is to be followed in the general case. Let Π^a be the auxiliary ring of an arbitrary grating Γ. Then we shall simply *define* the ideal Ω^a to be the one determined by the congruences (9:3), (9:4), and (9:5). The ideal Ω^a will evidently be a preferred one. For if we take any congruence belonging to any one of the three sets and identify either the τ-components or the boundaries

of its two members, we can immediately verify, by direct calculation, that the resulting congruence is a formal consequence of the original congruences of the set from which it was derived.

10. To make the ring of chains Π an effective working instrument, we shall want to be able to tell when two different polynomials of Π^a represent the same chain. The desired criterion will be obtained by defining a linear operator $(^N)$ reducing each polynomial K of Π^a to a *normal form* K^N,

$$K^N \equiv K \qquad \text{mod } \Omega^a, \tag{10:1}$$

such that two different normal polynomials will never be congruent, modulo Ω^a.

Given an arbitrary polynomial K of Π^a we shall first eliminate all the marks c_s with the aid of Relations (9:3), thereby reducing K to a linear combination K^c of monomials of the form

$$F = z_{s_1} z_{s_2} \cdots z_{s_n} \qquad (z_{s_i} = a_{s_i} \text{ or } b_{s_i}). \tag{10:2}$$

It will then be sufficient to define the normal forms of the monomials (10:2). The case $F = 1$ $(n = 0)$ will be disposed of by writing $F^N = F = 1$. In handling the remaining cases, it will make matters clearer if we first arrange the indices s associated with the covering pairs of Γ in some arbitrary linear order, which we shall call their *normal* order. Now, suppose the indices s_i and s_{i+1} of two consecutive factors of F are distinct $(s_i \neq s_{i+1})$, but not in normal order. Then, by (9:5), we may permute the two factors so as to bring their indices into normal order, provided we change the sign of the entire expression if both the factors are b_s's. Moreover, if the indices s_i and s_{i+1} are identical, $(s_i = s_{i+1})$, then, by (9:4), we may discard the factor z_{s_i} if it is an a_s, or reduce the whole expression to 0 if z_{s_i} is a b_s. Thus, by a finite succession of steps, we can reduce the monomial F to what we shall call its *normal form* F^N, where F^N will either vanish or be a product of factors with distinct indices arranged in normal order.

It is important to notice that the normal form F^N is independent of the order in which the steps of the reduction are carried out. Indeed, when we permute two factors in different indices we leave unchanged the sequential order of the factors in each separate index s. Moreover, when we reduce a sequence of factors in the same index s, the sequence invariably reduces to a_s if all its factors are of the form a_s, it reduces to b_s if its last factor is b_s and the others are of the form a_s, and it reduces to 0 if any factor except the last is b_s.

The normal form K^N of a general polynomial K will be obtained by reducing K to the form K^c, in which the c_s's have all been eliminated, by then reducing the terms of K^c to normal form and, finally, by collecting like normal terms.

There remains the question whether two normal polynomials K^N and L^N can be congruent, modulo Ω^a. Now, since the reduction of a polynomial to normal form is independent of the order in which the successive steps in the

reduction are performed, the formal identity $K^N = L^N$ necessarily implies the formal identities $(MKN)^N = (MLN)^N$. Therefore the polynomials K of normal form 0 will be the elements of an ideal Ω within the ring Π^a. If we can show that the ideal Ω contains (and, therefore, coincides with) the ideal Ω^a we can be sure that no two normal polynomials will be congruent, modulo Ω^a. But this last point can be settled, at once, by merely verifying that the determining polynomials of Ω^a (appearing in (9:3), (9:4), and (9:5) after transformation of the right-hand members) all have the normal form 0 and, therefore, all belong to Ω. Thus, the ideals Ω and Ω^a must surely be identical.

From this point on, we are going to discard the auxiliary ring Π^a in favor of the ring Π. Let us, therefore, agree to regard two polynomials K and L as formally identical

$$K = L$$

if they represent the same chain of Π^a,[8]

$$K \equiv L \qquad \text{mod } \Omega^a.$$

The elements K of Π will be called the *chains* of Π (and also of Γ).

11. We can now complete the identification of the chains of a coördinate grating Γ with the chains of the complex Θ associated with Γ (cf. §9).

THEOREM [11:1]. *Let Γ be any finite grating made up of n indexed pairs (a_s, c_s), $(s = 1, 2, \cdots, n)$. Then the chains K of Γ consist of all linear combinations of the 3^n chains of the form*

$$F = z_1 z_2 \cdots z_n \qquad (z_i = a_i, b_i, \text{ or } c_i). \tag{11:1}$$

Moreover, these last are linearly independent.

PROOF. Every chain K of Γ is a linear combination of normal monomials F^N. Moreover, by the introduction of suitable dummy symbols $x_s = 1$, the normal monomials F^N may evidently be written in the form

$$F^N = x_1 x_2 \cdots x_n \qquad (x_i = a_i, b_i, \text{ or } 1). \tag{11:2}$$

Therefore, to express a chain K as a linear combination of chains F we have only to express it as a linear combination of normal polynomials F^N and then to replace the dummy symbols $x_s = 1$, wherever they occur, by the equivalent expressions $a_s + c_s$ (cf. (9:3)). The linear independence of the chains F follows immediately from the known linear independence of the chains F^N. We merely observe that each of the two sets (11:1) and (11:2) is linearly dependent on the other and that each consists of precisely 3^n chains.

In the case where Γ is the coördinate grating of a space R^n we shall identify the chains (11:1) of Γ with the chains determined by the individual cells (9:2)

[8] It may be remarked that a similar convention is made in the theory of ordinary commutative polynomials, in which we identify a polynomial K with all polynomials obtainable from K by permuting the factors in the various terms of K.

of the complex Θ, thereby identifying the ring of chains of Γ with the ring of chains of Θ.

Now if Γ is an arbitrary grating and K an arbitrary chain of Γ there is always some finite approximation Γ^α of Γ to which the chain K also belongs. Suppose the covering pairs of Γ^α are (a_s, c_s), $(s = s_1, s_2, \cdots, s_n)$. Then, by what we have just shown above, there will be a unique way of expressing the chain K as a linear combination

$$K^\alpha = \sum_i \sigma_i F_i^\alpha \tag{11:3}$$

of chains of the form

$$F = z_{s_1} z_{s_2} \cdots z_{s_n} \qquad (z_{s_i} = a_{s_i}, b_{s_i} \text{ or } c_{s_i}). \tag{11:4}$$

We shall say that K^α is a *canonical form* of the chain K. There will thus be as many canonical forms K^α of K as there are finite approximating gratings Γ^α containing K. We shall call the chains F^α (11:4), of a finite grating Γ^α the *cells* of Γ^α. An infinite grating will not have cells.

12. Every preferred sub-ring Φ of the ring of chains Π will have a connectivity theory analogous, in every essential respect, to the classical theory of chains on a manifold. A chain [τ-chain] K of Φ will be called a *cycle* [τ-cycle] if its boundary vanishes; it will be said to bound, or to be *homologous* to 0,

$$K \sim 0,$$

if there exists a chain L of Φ such that its boundary is K,

$$L' = K.$$

We shall allow the terms of an homology to be transposed in the usual manner, so that $K \sim L$ will be equivalent to $K - L \sim 0$.

THEOREM [12:1]. *Every bounding chain* [τ-*chain*] *is a cycle* [τ-*cycle*].

Because $K = L'$ implies $K' = L'' = 0$, by (7:7).

COROLLARY [12:2]. *If two chains K and L are homologous, they have the same boundary.*

Because $K - L \sim 0$ implies $(K - L)' = K' - L' = 0$.

LEMMA [12:3]. *The components K^τ (and, therefore, also the conjugate K^*) of a cycle K are cycles.*

By [7:3] and (6:2).

LEMMA [12:4]. *Every linear combination of cycles* [τ-*cycles*; *bounding* τ-*cycles*] *is a cycle* [τ-*cycle*; *bounding* τ-*cycle*].

Since the bounding operator is linear.

THEOREM [12:5]. *The τ-cycles form a group Δ^τ with respect to addition; the bounding cycles form a subgroup Δ^τ of Δ^τ.*

By [12:4] and [12:1].

We shall call the group $\Xi^\tau = \Delta^\tau \bmod \Delta^\tau$ the τ^{th} *connectivity group* of the ring Φ.

LEMMA [12:6]. *The product KL of two cycles K and L is a cycle.*

Because $K' = L' = 0$ implies $(KL)' = K'L + K^*L' = 0$.

THEOREM [12:7]. *The cycles of Φ form a preferred sub-ring Δ of Φ.*

By [12:3], [12:4], and [12:6].

LEMMA [12:8]. *If L is a bounding cycle and K an arbitrary cycle, the products KL and LK are both bounding cycles.*

PROOF. Since K is a cycle, we have $K' = (K^*)' = 0$. Moreover, since L is a bounding cycle, there exists a chain M of Φ such that $M' = L$. Therefore, we have

$$(K^*M)' = (K^*)'M + KM' = KM' = KL,$$

and

$$(MK)' = M'K + M^*K' = M'K = LK,$$

thus establishing that KL and LK are bounding cycles.

THEOREM [12:9]. *The bounding cycles form a two-sided preferred ideal Λ within the ring Δ of all the cycles of Φ.*

By [12:3], [12:4], and [12:8]. *We do not assert that Λ is a preferred ideal of the ring Φ.*

We shall call the ring $\Xi = \Delta \bmod \Lambda$ the *connectivity ring* of the ring Φ. To be exact, the system Ξ is a somewhat stronger invariant than an ordinary ring, since its underlying group (with respect to the operation of addition) admits a well determined decomposition into the direct sum of the connectivity groups Ξ^τ $(\tau = 0, 1, 2, \cdots)$. If two connectivity rings are to represent the same invariant it is not enough that they be simply isomorphic. They must also be *completely isomorphic*, by which we shall mean that there exists a simple isomorphism between the two rings pairing the elements of the τ^{th} connectivity group of one ring with the elements of the τ^{th} connectivity group of the other, $(\tau = 0, 1, 2, \cdots)$. Since the product of two 0-cycles is a 0-cycle, we may regard the system Ξ^0 not merely as a group but as a sub-ring of the system Ξ.

13. There are a number of obvious but useful generalizations of the connectivity ring of Φ. Suppose, for example, we have two preferred subrings Φ and Ψ of Π. Then, as an immediate consequence of [12:8], the cycles of Φ which bound in Ψ will form a preferred ideal $\Lambda(\Phi, \Psi)$ within the ring $\Delta(\Phi)$ of all the cycles of Φ. We shall thus be able to form a ring $\Xi(\Phi, \Psi) = \Delta(\Phi) \bmod \Lambda(\Phi, \Psi)$, called the connectivity ring of Φ *relative to* Ψ.

Another important case is the one where we have a preferred sub-ring Φ and a preferred ideal Ω of Π. We shall say that a chain K of Π is a *cycle, modulo* Ω, if its boundary is in Ω,

$$K' \equiv 0 \qquad \bmod \Omega,$$

and that a cycle K, modulo Ω, is *homologous to* 0, modulo Ω,

$$K \sim 0 \qquad \bmod \Omega,$$

if there exists a chain L such that we have

$$L' \equiv K \qquad \text{mod } \Omega.$$

The entire discussion of §12 will obviously be valid if we operate with chains, cycles, and homologies, modulo Ω, in which case we shall be led to the *connectivity ring* $\Xi(\Phi \bmod \Omega)$ *of the ring* $\Phi \bmod \Omega$. In the classical theory, systematic use of concepts analogous to the above has been made by Lefschetz.

We shall now say a word about the system consisting of the ring of chains Π itself in combination with an arbitrary preferred ideal Ω of Π. This is the system which arises in connection with the topological duality theorem.

LEMMA [13:1]. *Let* Π *be the ring of chains of a grating* Γ. *Then every cycle* K *of* Π *is homologous to an integer. In other words, the connectivity ring* Ξ *of* Π *reduces to the ring* Ξ^0 *determined by the* 0*-chains and is simply isomorphic with the ring of the integers.*[9]

PROOF. Every cycle K of the ring Π is expressible as a linear combination (11:3) of cells of some finite approximation Γ^α of Γ. Moreover, if we write $F^\alpha = zG^\alpha$ ($z = a, b, c$; index not written) so as to exhibit the first factor z of each cell F^α of Γ^α we shall have

$$K = K^\alpha = \sum (\lambda_i a + \mu_i b + \nu_i c) G_i^\beta, \tag{13:1}$$

where, for the case $n > 1$, the expressions G_i^β will represent the cells of an approximating grating Γ^β with $n - 1$ indexed pairs, while, for $n = 1$, there will only be one term in the sum and we shall have $G_1^\beta = 1$.

To prove that K is homologous to an integer, we shall proceed by induction with respect to the rank n of the approximating grating Γ^α. If we subtract from K the boundary of the chain $\sum \mu_i a G_i^\beta$ we shall obtain a cycle

$$K_o = K - (\sum \mu_i a G_i^\beta)' = \sum (\lambda_i a + \nu_i c) G_i^\beta - \sum \mu_i a (G_i^\beta)'$$

homologous to K, which may evidently be reduced to the form

$$K_o = \sum (\rho_i a + \nu_i c) G_i^\beta, \tag{13:2}$$

since each of the expressions $(G_i^\beta)'$ is a linear combination of cells G_i^β. Moreover, the coefficients ρ_i and ν_i will have to be equal; otherwise the boundary of K_o would contain a non-vanishing term $(\rho_i - \nu_i) b G_i^\beta$, and K_0 would fail to be a cycle. In other words, K_o will have to be of the form

$$K_o = \sum \rho_i (a + c) G_i^\beta = \sum \rho_i G_i^\beta. \tag{13:3}$$

Thus, for the case $n = 1$, K_o will reduce to the integer ρ_1, while, for the case $n > 1$, it will reduce to a cycle of Γ^β. The cycle K ($K \sim K_o$) will, therefore, always be homologous to an integer, by the hypothesis of the induction.

LEMMA [13.2]. *If* Π *is the ring of chains of a grating* Γ *the only* 0*-cycles of* Π *are the integers.*

[9] Of course a 0-cycle never bounds, since there are no chains of type -1 for it to bound.

PROOF. Every 0-cycle of Π must be of the form (13:2), since it contains no symbols b_s. It may, therefore, be expressed in the form (13:3), whence, by induction, it must be reducible to an integer.

THEOREM [13:3]. *Let Π be the ring of chains of a grating Γ and let Ω be a preferred ideal of Π containing no integer chain of Π. Then the τ^{th} connectivity group of Π mod Ω is simply isomorphic with the $(\tau+1)^{\text{st}}$ connectivity group of Ω,*

$$\Omega^{\tau}(\Pi \bmod \Omega) \equiv \Xi^{\tau+1}(\Omega). \tag{13:4}$$

PROOF. A τ-cycle of Π mod Ω is a τ-chain K of Π with boundary in Ω. Therefore the boundary operator $(')$ transforms every τ-cycle K of Π mod Ω into a $(\tau+1)$-cycle K' of Ω. Moreover, every $(\tau+1)$-cycle of Ω is the transform K' of at least one τ-cycle K of Π mod Ω, since it bounds a cycle K of Π, by [13:1] and [13:2]. Thus, to establish the isomorphism (13:4) we have only to show that each of the two relations

$$K \sim 0 \qquad \text{mod } \Omega \tag{13:5}$$

and

$$K' \sim 0 \qquad \text{in } \Omega \tag{13:6}$$

implies the other. Now (13:5) implies the existence of a chain L of Π satisfying the relation $L' - K \equiv 0$, mod Ω; whence, $L' - K$ must belong to Ω; whence we must have $(L' - K)' = L'' - K' = K' \sim 0$ in Ω. On the other hand, (13:6) implies the existence of a chain M of Ω satisfying the relation $K' - M' = 0$; whence, $K - M$ must be a cycle of Π; whence, there must be a chain N of Π satisfying the relations $N' = K - M$; whence we must have $N' \equiv K \sim 0$, mod Ω.

COROLLARY [13:4]. *Let Ω be a preferred ideal of the type specified in* [13:3]. *Then the product of two cycles of Ω is always homologous to zero in Ω.*

PROOF. We have seen that the cycles of Ω are the boundaries K' of chains K of Π. Therefore, if K' and L' are any two cycles of Ω we shall have

$$(KL')' = K'L' + K^*L'' = K'L' \sim 0 \qquad \text{in } \Pi.$$

Moreover, the chain KL' will belong to Ω, since L' belongs to Ω and since Ω is an ideal. Therefore we shall also have $K'L' \sim 0$ in Ω.

REMARK. Theorems [13:3] and [13:4] obviously remain valid if Π is *any* ring of chains such that: (α) Every cycle K of Π is homologous to an integer, and (β) The only 0-cycles of Π are the integers.

14. With each chain K of a grating Γ we are now going to associate a space $|K|$, called the *locus* of K, which will be defined as follows. To simplify matters we shall choose our notations in such a way that the space on which the grating Γ is constructed is the unit element 1 of the underlying boolean algebra.[10]

[10] This is evidently always possible, since every sub-space x of 1 is the unit element of a sub-algebra consisting of all sub-spaces dominated by x.

Then the loci of the chains 0, 1, a_s, b_s, and c_s will be defined to be the spaces represented by the symbols themselves. Moreover, the locus of a canonical monomial

$$F = \lambda z_{s_1} z_{s_2} \cdots z_{s_n}, \qquad (z_{s_i} = a_{s_i}, b_{s_i}, \text{ or } c_{s_i}; \lambda \neq 0) \tag{14:1}$$

(i.e. a monomial *without repeated indices*, in the canonical form (11:4)) will be defined to be the intersection of the loci of the factors of the monomial,

$$| F | = z_{s_1} \frown z_{s_2} \frown \cdots \frown z_{s_n}. \tag{14:2}$$

Finally, the locus of an arbitrary chain K, *written in canonical form*

$$K = K^\alpha = \sum \sigma_i F_i^\alpha, \tag{14:3}$$

will be defined to be the union of the loci of the term of K,

$$K = | \sigma_1 F_1^\alpha | \smile | \sigma_2 F_2^\alpha | \smile \cdots \smile | \sigma_n F_n^\alpha |. \tag{14:4}$$

Since a chain K can usually be expressed in more than one canonical form, the following theorem will be required to make the definition of the locus $| K |$ determinate.

THEOREM [14:1]. *Two different canonical forms K^α and K^β of K determine the same locus $| K |$.*

PROOF. Let K^α and K^β belong to the finite approximating gratings Γ^α and Γ^β respectively. Then there is a third canonical form $K^{\alpha \cup \beta}$ belonging to the union $\Gamma^{\alpha \cup \beta} = \Gamma^\alpha \smile \Gamma^\beta$ of the two gratings. Thus, since we can pass from either of the gratings Γ^α or Γ^β to the grating $\Gamma^{\alpha \cup \beta}$ by the process of adjoining one new indexed pair at a time, the theorem reduces to the special case where Γ^β can be obtained from Γ^α by the adjunction of a single new pair (a, c), (index not written). But, in this simple case, the theorem is immediate. For the relation $a \smile c = 1$ implies, at once,

$$| F_i^\alpha | = | aF_i^\alpha | \smile | cF_i^\alpha |;$$

whence,

$$| K | = | \sum \lambda_i F_i^\alpha | = | \sum \lambda_i (a + c) F_i^\alpha | = | \sum \mu_j F_j^\beta |.$$

THEOREM [14:2]. *The locus of the sum [difference] of two chains K and L is dominated by the union of the loci of K and L,*

$$| K \pm L | \subseteq | K | \smile | L |. \tag{14:5}$$

This may at once be seen by expressing K and L in canonical form with respect to the same approximating grating Γ^α.

THEOREM [14:3]. *The locus of the product of two chains K and L is dominated by the intersection of the loci of K and L*

$$| KL | \subseteq | K | \frown | L |. \tag{14:6}$$

PROOF. The theorem obviously reduces to the case where K and L are cells of the same approximating grating Γ^a,

$$K = z_{s_1} z_{s_2} \cdots z_{s_n}, \qquad L = w_{s_1} w_{s_2} \cdots w_{s_n}, \qquad (z_{s_i}, w_{s_i} = a_{s_i}, b_{s_i}, \text{ or } c_{s_i}).$$

Now, in this special case, the product KL either vanishes or is of the form

$$KL = u_{s_1} u_{s_2} \cdots u_{s_n} \qquad (u_{s_i} = a_{s_i}, b_{s_i}, \text{ or } c_{s_i}),$$

where, by (9:4) and (9:4′), we have $|u_{s_i}| = |z_{s_i}| \frown |w_{s_i}|$. Relation (14:6) therefore follows at once.

THEOREM [14:4]. *The locus of a chain K is the union of the loci of its components*

$$|K| = |K^0| \smile |K^1| \smile |K^2| \smile \cdots, \tag{14:7}$$

whence, in particular,

$$|K^r| \subseteq |K|. \tag{14:8}$$

This is obvious if we express K in canonical form.

THEOREM [14:5]. *The locus of the boundary of a chain K is dominated by the locus of K.*

$$|K'| \subseteq |K|. \tag{14:9}$$

PROOF. The theorem again reduces to the case where K is a canonical monomial. The proof is then an immediate consequence of (7:5).

THEOREM [14:6]. *The set of all chains K of a grating Γ such that their loci are dominated by a space x is a preferred ideal Ω_x of Γ.*

By [14:2], [14:3], [14:4], and [14:5].

We shall say that a chain K of Γ is *empty* if its locus is 0, i.e. if it belongs to the ideal Ω_0 determined by the element 0. The connectivity ring Ξ ($\Pi \bmod \Omega_0$) of the set of all chains of Γ, modulo the empty chains is an important invariant of the grating Γ. We shall call it the *connectivity ring* $\Xi(\Gamma)$ of Γ,

$$\Xi(\Gamma) = \Xi(\Pi \bmod \Omega_0). \tag{14:10}$$

15. Now, let Γ be a grating of the space 1, and let the indexed pairs of Γ be (a_s, c_s). Then if x is any subspace of 1, we shall say that the intersection operator $(^x)$, §2, transforms the grating Γ of 1 into the grating Γ^x of x made up of the indexed pairs (a_s^x, c_s^x),

$$a_s^x = a_s \frown x, \qquad c_s^x = c_s \frown x.$$

Moreover, we shall say that $(^x)$ transforms a chain $K = \phi(a_s, b_s, c_s)$ of Γ into the chain $K^x = \phi(a_s^x, b_s^x, c_s^x)$ of Γ^x.

THEOREM [15:1]. *Let (x, y) be any covering pair of the space 1, let Π be the ring of chains of any grating Γ of 1, and let Ω_y be the preferred ideal of Π composed of all chains of Π such that their loci are dominated by y. Then, if Π^x and Ω_y^x are*

the intersections with x of Π and Ω_y respectively, the rings Ξ (Π mod Ω_y) and Ξ (Π mod Ω_y^x) will be completely isomorphic,

$$\Xi\ (\Pi \bmod \Omega_y) \equiv \Xi\ (\Pi \bmod \Omega_y^x). \tag{15:1}$$

PROOF. Since the operator $(^x)$ transforms the chains K of Π linearly into the chains K^x of Π^x, and since every chain K^x of Π^x is the transform of a definite chain K of Π, the problem reduces to showing that the two relations

$$K \equiv 0 \qquad \bmod \Omega_y \tag{15:2}$$

and

$$K^x \equiv 0 \qquad \bmod \Omega_y^x \tag{15:3}$$

are equivalent. Now (15:2) is merely the condition that the locus $|K|$ of K shall not meet the complement z of y, while (15:3) is the condition that the locus $|K^x|$ of the intersection of K with x shall not meet z. Furthermore, the space x dominates the space z, since (x, y) is a covering pair. Therefore the locus $|K|$ will meet z if, and only if, the locus $|K^x|$ does.

16. The problem of evaluating the connectivity ring $\Xi(\Gamma) = (\Pi \bmod \Omega_o)$ of a grating Γ can often be simplified by substituting for Γ a suitable approximating grating Γ_1. An approximation Γ_1 of Γ will be called a *basic approximation* provided that to every chain K of Γ such that the boundary of K is congruent, modulo Ω_o, to a chain of Γ_1 there corresponds a chain K_1 of Γ_1 such that we have $K_1 \sim K$, mod Ω_o. This condition evidently implies:

(i) That if a chain K of Γ is a cycle, mod Ω_o, it is homologous, mod Ω_o, to a cycle K_1 of Γ_1, mod Ω_o.

(ii) That every cycle L_1 of Γ_1, mod Ω_o, which bounds a chain K of Γ, mod Ω_o, also bounds a chain K_1 of Γ_1, mod Ω_o.

THEOREM [16:1]. *If Γ_1 is a basic approximation of Γ its connectivity ring is completely isomorphic with the connectivity ring of Γ,*

$$\Xi(\Gamma_1) \equiv \Xi(\Gamma). \tag{16:1}$$

PROOF. If we operate modulo Ω_o every element of $\Xi(\Gamma)$ is a maximal class $[K]$ of cycles of Γ homologous in Γ, every element of $\Xi(\Gamma_1)$ a maximal class $[K_1]$ of cycles of Γ_1 homologous in Γ_1. Therefore:

(α) Every element $[K_1]$ is a subclass of an element $[K]$. (Obviously.)

(β) Every element $[K]$ includes at least one subclass $[K_1]$, (by (i) above).

(γ) No element $[K]$ includes more than one subclass $[K_1]$. For if L_1 is any cycle of $[K]$ belonging to Γ_1, and K_1 any element of the known subclass $[K_1]$ of $[K]$, we must have $L_1 - K_1 \sim 0$, mod Ω_o, in Γ, by the very definition of $[K]$. Therefore, by (ii) above, we must also have $L_1 - K_1 \sim 0$, mod Ω_o, in Γ_1. Therefore, finally, L_1 must belong to the subclass $[K_1]$.

The correspondence associating each class $[K]$ with its well-determined subclass $[K_1]$ sets up the desired isomorphism between the rings $\Xi(\Gamma_1)$ and $\Xi(\Gamma)$.

THEOREM [16:2]. *If Γ_1 is a basic approximation of Γ and Γ_2 a basic approximation of Γ_1, then Γ_2 is a basic approximation of Γ.*

This is an immediate consequence of the definition of a basic approximation.

LEMMA [16:3]. *A sufficient condition that Γ_1 be a basic approximation of Γ is that to every finite approximation Γ^α of Γ there correspond a finite approximation Γ_1^β of Γ_1 such that Γ_1^β be a basic approximation of $\Gamma^\alpha \smile \Gamma_1^\beta$.*

Because of the fact that every chain K of Γ belongs to some finite approximation Γ^α of Γ. (The condition stated in the lemma is also necessary, but we shall only require its sufficiency.)

LEMMA [16:4]. *Let Γ^α be any finite grating with the cells F_i^α and let Γ^β be a refinement of Γ^α formed by adjoining to Γ^α a single new indexed pair (a, c), (index not written). Then Γ^α will be a basic approximation of Γ^β provided the locus of a cell F_i^α never meets the two members a and c of the pair (a, c) unless it also meets the barrier b; or, in other words, provided each cell F_i^α satisfies one, at least, of the following three conditions:*

$$bF_i^\alpha \not\equiv 0, \qquad aF_i^\alpha \equiv 0, \qquad cF_i^\alpha \equiv 0, \qquad \mod \Omega_o . \tag{16:2}$$

PROOF. We shall prove that when we operate modulo Ω_o every chain K^β of Γ^β with boundary congruent to a chain of Γ^α is homologous to a chain K^α of Γ^α. Let us write the chain K^β in canonical form,

$$K^\beta = \sum (\lambda_i a + \mu_i b + \nu_i c) F_i^\alpha . \tag{16:3}$$

Then the desired chain K^α may be chosen to be the following one:

$$\begin{aligned} K^\alpha &= K^\beta - (\textstyle\sum \mu_i a F_i^\alpha)' \\ &= \textstyle\sum (\rho_i a + \nu_i c) F_i^\alpha ; \end{aligned}$$

(cf. the proof of [13:1]). The chain K^α will obviously be homologous to K^β, since $K^\alpha - K^\beta$ is a boundary. Moreover, it will reduce, modulo Ω_o, to a chain of Γ^α, as we shall see by examining each of its terms separately.

(i) Suppose that for the cell F_i^α appearing in a term $(\rho_i a + \nu_i c)F_j^\alpha$ we have $bF_i^\alpha \not\equiv 0$. Then we must have $\rho_i = \nu_i$, otherwise the boundary of K^α would contain a non-empty term $(\rho_i - \nu_i)bF_i^\alpha$ and would, therefore, not be congruent to a chain of Γ^α. Therefore

$$(\rho_i a + \nu_i c)F_i^\alpha = \rho_i(a + c)F_i^\alpha = \rho_i F_i^\alpha .$$

(ii) Suppose we have $aF_i^\alpha = 0$. Then

$$(\rho_i a + \nu_i c)F_i^\alpha \equiv \nu_i(a + c)F_i^\alpha = \nu_i F_i^\alpha ,$$

(iii) Suppose we have $cF_i^\alpha = 0$. Then

$$(\rho_i a + \nu_i c)F_i^\alpha \equiv \rho_i(a + c)F_i^\alpha - \rho_i F_i^\alpha .$$

The chain K^α therefore reduces to a linear combination of cells of Γ^α, whence it follows that Γ^α is a basic approximation of Γ^β.

THEOREM [16:5]. *A sufficient condition that an approximation* Γ_1 *of a grating* Γ *be a basic approximation is that to each indexed pair* (a_s, c_s) *of* Γ *there correspond a finite approximation* Γ_s *of* Γ_1 *such that every 0-cell of* Γ_s *be dominated either by* a_s *or by* c_s.

PROOF. We shall show that when we operate modulo Ω_o every chain K of Γ such that its boundary is congruent to a chain K_1 of Γ_1 is homologous to a chain of Γ_1. Let us measure the *remoteness* of K from Γ_1 by the number of distinct indices appearing in K which are not associated with the indexed pairs of Γ_1. Then, if the remoteness of K from Γ_1 is 0, we may, at once, write $K_1 = K$, without further argument. The case where the remoteness of K is an integer n $(n > 0)$ will be treated by induction with respect to n. Let Γ^α be any finite approximation of Γ containing K, and let (a_s, c_s) be any indexed pair of Γ^α but not of Γ_1. Then, under the condition stated in the theorem, there exists a finite approximation Γ_1^β of Γ_1 such that every 0-cell (and, therefore, also every cell) of Γ_1^β is dominated either by a_s or by c_s. Now suppose we form the union $\Gamma^{\alpha\cup\beta} = \Gamma^\alpha \cup \Gamma_1^\beta$ of the two gratings Γ^α and Γ_1^β and the approximation Γ^γ of $\Gamma^{\alpha\cup\beta}$ obtained by omitting the pair (a_s, c_s). Then, since Γ^γ is a refinement of Γ_1^β every cell F_i^γ of Γ^γ must be dominated, *a fortiori*, either by a_s or by c_s, from which it obviously follows that we have $|\, b_s F_i^\gamma \,| = |\, a_s F_i^\gamma \,|$ or $|\, c_s F_i^\gamma \,|$. Therefore, F_i^γ cannot meet both a_s and c_s without meeting b_s. In other words, Γ_γ must be a basic approximation of $\Gamma^{\alpha\cup\beta}$, since it fulfils the conditions of [16:4]. We therefore conclude that the chain K (which belongs to Γ^α and therefore to $\Gamma^{\alpha\cup\beta}$) is homologous, modulo Ω_o, to a chain of Γ^γ of remoteness no greater than $n - 1$ from Γ_1. The chain K is, therefore, homologous, modulo Ω, to a chain of Γ_1, by the hypothesis of the induction.

COROLLARY [16:6]. *A sufficient condition that an approximation* Γ_1 *of a grating* Γ *be a basic approximation is that to each indexed pair* (a, c) *of* Γ *there correspond an indexed pair* (a_1, c_1) *of* Γ_1 *such that we have either*

$$a \supseteq a_1, \qquad c \supseteq c_1,$$

or

$$a \supseteq c_1, \qquad c \supseteq a_1.$$

For, in this case, the approximating grating consisting of the single pair (a_1, c_1) fulfils the conditions of the theorem.

COROLLARY [16:7]. *If* Γ *is a singular grating with repeated covering pairs, the non-singular grating* Γ_1 *formed by discarding all but one pair from each set of replicas, is a basic approximation of* Γ.

This is an immediate consequence of [16:6]. We notice, further, that whenever the grating Γ contains two pairs (a_s, c_s) and (a_t, c_t) such that $a_s = c_t$ and $c_s = a_t$ we may also discard either of the two pairs.

17. The covering pairs of a grating Γ determine certain associated covering pairs obtainable in the following manner. We take any finite approximation

Γ^α of Γ and arrange the 0-cells in two arbitrary classes A and C such that each 0-cell belongs to at least one of the two classes and, possibly, to both. Then if a is the union of the loci of the 0-cells belonging to A, and c the union of the loci of the 0-cells belonging to C, the pair (a, c) is evidently a covering pair of the space containing the grating Γ. We shall say that (a, c) is a pair *implied* by the grating Γ.

THEOREM [17:1]. *An approximation Γ_1 of Γ is a basic approximation of Γ provided every indexed pair (a, c) of Γ is a pair implied by Γ_1.*

For suppose a and c are constructed out of the 0-cells of a finite approximation Γ_1^α of Γ_1. Then every 0-cell of Γ_1^α is covered by one, at least, of the members a and c of (a, c). Therefore, the theorem is a consequence of [16:5].

18. We shall now prove a useful lemma about singular gratings. Let Γ be a singular grating made up of certain indexed pairs (a_s, c_s), and let $s \to t$ be a single-valued transformation of the indices with the property that if (a_s, c_s) is any covering pair of Γ then the pair (a_t, c_t) determined by the transformation $s \to t$ is a replica of (a_s, c_s). Then if $K = \phi(a_s, b_s, c_s)$ is an arbitrary chain of Γ we shall say that the chain $L = \phi(a_t, b_t, c_t)$ determined by the transformation $s \to t$ is a *replica* of K. The replicas of a cycle will obviously all be cycles. Moreover, if Ω is any *replica-containing* preferred ideal (i.e., a preferred ideal such that the replicas of every chain of the ideal are in the ideal), the replicas of a cycle, modulo Ω, will obviously be cycles modulo Ω. The ideal Ω_0 of the empty chains is a typical example of a replica-containing preferred ideal.

LEMMA [18:1]. *Let Γ be a singular grating, let Π be the ring of chains of Γ, and let Ω be a replica-containing preferred ideal of Π. Then if K is a cycle of Π modulo Ω, every replica L of K is homologous to K, modulo Ω.*

The theorem may be proved by induction with respect to the number of distinct indices n appearing in K. For the case $n = 0$, we shall have $K = L =$ const. For the case $n > 0$, we shall write

$$K = \sum (\lambda_i a + \mu_i b + \nu_i c) F_i^\alpha ,$$

(cf. [16:3]), whence

$$L = \sum (\lambda_i \alpha + \mu_i \beta + \nu_i \gamma) \Phi_i^\alpha ,$$

where α, β, γ, and Φ_i^α are replicas of a, b, c, and F_i^α respectively. According to the argument in [16:3], we shall now have

$$K \sim K - (\sum \mu_i b F_i^\alpha)' \equiv \sum \sigma_i F_i^\alpha , \qquad \text{mod } \Omega;$$

$$L \sim L - (\sum \mu_i \beta \Phi_i^\alpha)' \equiv \sum \sigma_i \Phi_i^\alpha , \qquad \text{mod } \Omega.$$

Moreover, by the hypothesis of the induction,

$$\sum \sigma_i F_i^\alpha \sim \sum \sigma_i \Phi_i^\alpha , \qquad \text{mod } \Omega.$$

Therefore, finally,

$$K \sim L, \qquad \text{mod } \Omega.$$

REMARK. In defining a replica L of a chain K we are not assuming that the inverse of the transformation $s \rightarrow t$ is single-valued in the indices. The replica L may, therefore, be *degenerate*, in the sense that two or more indices of K may be transformed into the same index of L. If L is a degenerate replica of K, K is not a replica of L.

19. A covering pair (a, c) will be said to *dominate* a covering pair (a_1, c_1) if the spaces a and c dominate the spaces a_1 and c_1 respectively. A grating Γ will be said to *dominate* a grating Γ_1 if there is a one-one correspondence between the indexed pairs (a_s, c_s) of Γ and the indexed pairs (a_{s_1}, c_{s_1}) of Γ_1 such that each pair (a_s, c_s) dominates the corresponding pairs (a_{s_1}, c_{s_1}). If Γ dominates Γ_1, we shall say that each chain $K = \phi(a_s, b_s, c_s)$ of Γ dominates the corresponding chain $K_1 = \phi(a_{s_1}, b_{s_1}, c_{s_1})$ of Γ_1.

THEOREM [19:1]. *If Γ dominates Γ_1, the congruence $K \equiv 0$, mod Ω_o, evidently implies the congruence $K_1 \equiv 0$, mod Ω_o; whence, if K is a cycle [a bounding cycle], mod Ω_o, so also is K_1.*

A grating Γ will be said to dominate a grating Γ_1 *sharply* if it not only dominates the grating Γ_1 but has the further property that $K_1 \equiv 0$, mod Ω_o, implies $K \equiv 0$, mod Ω_o.

THEOREM [19:2]. *If Γ dominates Γ_1 sharply, the connectivity rings of the two gratings are completely isomorphic,*

$$\Xi(\Gamma) \equiv \Xi(\Gamma_1). \tag{19:1}$$

Because the cycles [bounding cycles], mod Ω_o, of one grating correspond to the cycles [bounding cycles], mod Ω_o, of the other.

20. In applying the theory of gratings to a general space x, the two most plausible gratings with which to operate are: (i) the grating Γ_p consisting of all covering pairs (a_p, c_p) such that a_p and c_p are both closed; (ii) the grating Γ_q consisting of all covering pairs (a_q, c_q) such that a_q and c_q are both open.[11] We shall call the covering pairs of Γ_p[of Γ_q] the *closed* [*open*] *covering pairs* of x, and the chains of Γ_p[of Γ_q] the *closed* [*open*] *chains* of x.[12] Moreover, we shall say that the rings $\Xi(\Gamma_p)$ and $\Xi(\Gamma_q)$ are the *closed* and *open connectivity rings* of the space respectively. For a general space x, these two rings will be independent invariants. However, we shall prove that if the space x is normal the rings will be completely isomorphic with one another.

[11] The grating Γ_r consisting of all covering pairs (a_r, c_r), such that one member is closed and the other open, appears to be of no particular interest. For, by [16:6], there is a basic approximation Γ_s of Γ_r consisting of all covering pairs of Γ_r such that one member is the complement of the other. Moreover, since the barriers of Γ_s are all empty, every chain, modulo Ω_o, of Γ_s reduces to a 0-chain.

[12] The terms 'closed' and 'open,' as applied to chains, are sometimes used in quite another connection to indicate whether or not a chain is a cycle. To avoid confusion, we shall only use them in the sense indicated in the text.

LEMMA [20:1]. *Let x be an arbitrary normal space. Then every open covering pair (a_q , c_q) of x dominates some closed covering pair (a_p , c_p).*

PROOF. The lemma is merely a restatement, in dual form, of the condition that x be a normal space. The complements α_p and γ_p of a_q and c_q are closed spaces with the intersection 0; the spaces a_p and c_p are the complements of two open spaces α_q and γ_q with the intersection 0 and such that α_q and γ_q dominate α_p and γ_p respectively.

THEOREM [20:2]. *Let x be a normal space, let Γ_p and Γ_q be the gratings of the closed and of the open covering pairs of x respectively, and let $\Gamma_{p\cup q}$ be the unions of the gratings Γ_p and Γ_q. Then the connectivity rings of Γ_p and $\Gamma_{p\cup q}$ are completely isomorphic,*

$$\Xi(\Gamma_p) \equiv \Xi(\Gamma_{p\cup q}). \tag{20:1}$$

For, by [20:1] and [16:6], Γ_p is a basic approximation of $\Gamma_{p\cup q}$.

Next, we shall prove our assertion that if x is a normal space the rings $\Xi(\Gamma_p)$ and $\Xi(\Gamma_q)$ are completely isomorphic. As a first step we shall replace the gratings Γ_p and Γ_q by two singular gratings Γ_p^* and Γ_q^* made up of replicas of the covering pairs of Γ_p and of Γ_q respectively. Let us consider all possible couples

$$(a_p , c_p):(a_q , c_q), \qquad (a_p \subseteq a_q , c_p \subseteq c_q), \tag{20:2}$$

involving an arbitrary closed covering pair (a_p , c_p) in association with an arbitrary open covering pair (a_q , c_q) dominating (a_p , c_p), and let us assign a distinguishing symbol s to every such couple. Then we shall define Γ_p^* and Γ_q^* by saying that each couple s determines an indexed replica of its first member (a_p , c_p) belonging to Γ_p^* and an indexed replica of its second member (a_q , c_q) belonging to Γ_q^*. Moreover, we shall assign to the two replicas the index s associated with their determining couple. The grating Γ_p^* will thus contain at least one replica of every closed covering pair (a_p , c_p), since we can always form a couple (20:2) consisting of the pair (a_p , c_p) and of the open covering pair (1, 1). Similarly the grating Γ_q^* will contain at least one replica of every open covering pair (a_q , c_q), since, by [20:1], there is at least one closed covering pair dominated by (a_q , c_q) with which (a_q , c_q) may be coupled. Therefore, by [16:7], we shall have the relations $\Xi(\Gamma_p) \equiv \Xi(\Gamma_p^*)$ and $\Xi(\Gamma_q) \equiv \Xi(\Gamma_q^*)$, so that the problem will reduce to proving the relation $\Xi(\Gamma_p^*) \equiv \Xi(\Gamma_q^*)$.

Now, the grating Γ_q^* clearly dominates the grating Γ_p^*, since the s^{th} covering pair of Γ_q^* dominates the s^{th} covering pair of Γ_p^*. To avoid ambiguity we shall say that a chain K_q of Γ_q^* *s-dominates* that particular chain K_p of Γ_p^* with which it is associated when we couple the covering pairs of Γ_p^* and Γ_q^* of like index. (The chain K_q may also dominate, though it will not s-dominate, other chains of Γ_p^* corresponding to other ways of coupling the covering pair of the two gratings). Let us now make the following observations. Every cycle [bounding cycle] K_q, mod Ω_o, of Γ_q^* s-dominates a cycle [bounding cycle] K_p, mod Ω_o,

of Γ_p^*. Therefore, if it were always true that the chain K_q s-dominating a cycle [bounding cycle] K_p, mod Ω_o, of Γ_p^* was a cycle [bounding cycle], mod Ω_o, of Γ_q^*, the desired isomorphism would follow without further argument. Since this is not the case, we shall be reduced to proving that among the replicas of a cycle [bounding cycle] K_p, mod Ω_o, of Γ_p^* there is at least one which is dominated by a cycle [bounding cycle], mod Ω_o, of Γ_q^*. This will be sufficient to establish the desired isomorphism, since we have shown, [18:1], that every replica of a cycle K, mod Ω_o, (whether of Γ_p^* or of Γ_q^*) belongs to the same homology class as K. We shall first prove three easy lemmas.

LEMMA [20:3].[13] *Let x be an arbitrary normal space. Then to every finite set of closed subspaces p_i $(i = 1, 2, \cdots, n)$ of x with the intersection* 0,

$$p_1 \frown p_2 \frown \cdots \frown p_n = 0, \tag{20:3}$$

there corresponds a set of open subspaces q_i $(i = 1, 2, \cdots, n)$, also with the intersection 0,

$$q_1 \frown q_2 \frown \cdots \frown q_n = 0, \tag{20:4}$$

such that each subspace q_i dominates the corresponding subspace p_i.

PROOF. The case $n = 1$ is trivial; the space p_1 must necessarily be the space 0, and we may, therefore, write $q_1 = 0$. The case $n > 0$ follows, immediately, by induction with respect to n. Let us denote the intersection of the first $n - 1$ subspaces p_i by $p = p_1 \frown p_2 \frown \cdots \frown p_{n-1}$, so that (20:3) may be written $p \frown p_n = 0$. Then we shall have two subcases to consider, according as p is, or is not, the space 0. If we have $p = 0$, we know, by the hypothesis of the induction, that there exist open spaces q_i $(i < n)$ dominating the spaces p_i respectively and satisfying the relation $q_1 \frown q_2 \frown \cdots \frown q_{n-1} = 0$. Consequently, to complete the determination of the q_i's, we have only to write $q_n = 1$. If we have $p \neq 0$, we know, since the space is normal, that there exist open subspaces q and q_n dominating p and p_n respectively, and such that we have $q \frown q_n = 0$. Therefore we may complete the determination of the q_i's by merely writing $q_i = q$ $(i < n)$. This completes the proof of the first lemma.

LEMMA [20:4]. *Let x be an arbitrary normal space. Then to every finite set of closed subspaces p_i $(i = 1, 2, \cdots, n)$ of x there corresponds a set of open subspaces q_i $(i = 1, 2, \cdots, n)$ dominating the spaces p_i respectively and such that for every subset of the p_i's with the intersection* 0,

$$p_{s_1} \frown p_{s_2} \frown \cdots p_{s_m} = 0, \tag{20:5}$$

the corresponding subset of the q_i's also has the intersection 0,

$$q_{s_1} \frown q_{s_2} \frown \cdots \frown q_{s_m} = 0. \tag{20:6}$$

PROOF. There are, in all, a finite number of relations (20:5). Let us number them with the indices $\alpha = 1, 2, \cdots, h$. Then, by [20:3], corresponding to the

[13] This and the next lemma are proved in Alexandroff and Hopf, *Topologie*, p. 71 et seq.

α^{th} relation (20:5) there will be a set of open subspaces $q^{\alpha}_{s_i}$ dominating the corresponding spaces p_{s_i} and having the intersection 0,

$$q^{\alpha}_{s_1} \frown q^{\alpha}_{s_2} \frown \cdots \frown q^{\alpha}_{s_m} = 0. \tag{20:7}$$

Now, for any index s not appearing in (20:7), let us write $q^{\alpha}_{s} = 1$, so as to complete the determinations of the q^{α}_{s}'s. Then, the desired subspaces q_i may be obtained by merely writing

$$q_i = q^1_i \frown q^2_i \frown \cdots q^h_i.$$

The space q_i will obviously dominate the space p_i, since each q^{α}_i dominates p_i. Moreover, Relations (20:7) will imply Relations (20:6).

LEMMA [20:5]. *Every finite approximation Γ^0_p of Γ^*_p is sharply dominated by some finite approximation Γ^1_q of Γ^*_q. [Notice that, in general, the grating Γ^1_q is not the one with which Γ^0_p is coupled when we associate the covering pairs of Γ^*_p with the similarly indexed pairs of Γ^*_q.]*

For suppose we denote the covering pairs of Γ^0_p by (a^0_p, c^0_p), and the covering pairs of Γ^1_q by (a^1_q, c^1_q). Then, by [20:4], the open sets a^1_q and c^1_q may be so chosen that whenever a cell of Γ^0_p is empty, so also is the corresponding cell of Γ^1_q.

On the basis of [20:5] we can, at once, prove the last outstanding point: that if K_p is a cycle [bounding cycle], mod Ω_o, of Γ^*_p there is always a replica L_p of K_p which is s-dominated by a cycle [bounding cycle] K_q, mod Ω_o, of Γ^*_p. Indeed, every cycle K_p of Γ^*_p belongs to some finite approximation Γ^0_p of Γ^*_p sharply dominated (in the notation of [20:5]) by a finite approximation Γ^1_q of Γ^*_q. Let the correspondence between the covering pairs of Γ^0_p and Γ^1_q be

$$(a^0_p, c^0_p) : (a^1_q, c^1_q). \tag{20:8}$$

Then every couple appearing in (20:8) also appears in (20:2) where it is associated with a definite index s. Now suppose we merely re-index the pairs (a^0_p, c^0_p) of Γ^0_p and the pairs (a^1_p, c^1_p) of Γ^1_p so that they shall both have the index s. Then the effect of this change of indices will be to transform the grating Γ^0_p into a replica Γ^2_p of itself and the grating Γ^1_q into a replica Γ^2_q of itself, such that the covering pair of index s in Γ^2_q sharply dominates the covering pair of the same index s in Γ^2_p. The replica in Γ^2_p of the cycle K_p in Γ^0_p will be the desired replica L_p. For it will be sharply s-dominated by the corresponding chain of Γ^2_q, which last will also be a cycle, mod Ω_o.

The case where K_p is a bounding cycle, mod Ω_o, can be treated in a similar manner. If K_p bounds, there is a chain M_p such that we have $M'_p \equiv K_p$, mod Ω_o; the chain M_p belongs to a finite approximation Γ^0_p of Γ^*_p; to Γ^0_p there corresponds a finite approximation Γ^1_q of Γ^*_q, and so on. The isomorphism $\Xi(\Gamma^*_p) = \Xi(\Gamma^*_q)$ is, therefore, established; whence, also, the desired theorem:

THEOREM [20:6]. *Let x be any arbitrary normal space. Then the closed and open connectivity rings of x are completely isomorphic*:

$$\Xi(\Gamma_q) = \Xi(\Gamma_p). \tag{20:7}$$

21. We now observe that the entire argument of §20 will continue to be valid if we replace the grating Γ_p by an arbitrary grating Γ_p^1 of closed, indexed covering pairs, such that every open covering pair dominates at least one covering pair of Γ_p^1. The connectivity ring of any such grating Γ_p^1 will, therefore, be completely isomorphic with the closed connectivity ring of the underlying space

$$\Xi(\Gamma_p^1) \equiv \Xi(\Gamma_q) \equiv \Xi(\Gamma_p). \tag{21:1}$$

Given a normal space x, a grating Γ_p^0 made up of closed, indexed covering pairs of x will be called a *fundamental grating* provided that if α_p and γ_p are any two closed subspaces of x with the intersection 0 there exists a finite approximation of Γ_p^f of Γ_p^0 such that no 0-cell of Γ_p^f meets both α_p and γ_p.

THEOREM [21:1]. *Let x be a normal space, and let Γ_p^0 be a fundamental grating of x. Then the connectivity ring of Γ_p^0 is completely isomorphic with the closed connectivity ring of x,*

$$\Xi(\Gamma_p^0) = \Xi(\Gamma_p).$$

PROOF. Let Γ_p^1 be the grating made up of all the covering pairs implied by Γ_p^0 (cf. §17). Then, by [17:1], we shall have

$$\Xi(\Gamma_p^1) \equiv \Xi(\Gamma_p^0).$$

Therefore it will be sufficient to show that the connectivity ring of Γ_p^1 satisfies Relation (21:1). Let (a_q, c_q) be any open covering pair of x. Then the complements α_p and γ_p of a_q and c_q respectively are closed sets with the intersection 0. Hence, by hypothesis, there exists a finite approximation Γ_p^f of Γ_p^0 such that no 0-cell of Γ_p^f meets both α_p and γ_p. Now, let a_p be the union of the loci of the 0-cells of Γ_p^f which do not meet α_p, and let c_p be the union of the loci of the 0-cells which do not meet γ_p. Then (a_p, c_p) is a covering pair implied by Γ_p^0. Moreover (a_p, c_p) is dominated by (a_q, c_q). Therefore, finally, the grating Γ_p^1 satisfies Relation (21:1).

ILLUSTRATION. Let R be the point space of a (finite or transfinite) set of real variables ξ_i. Moreover, let the topology of the space R be determined by the conditions that a point P_0 shall be a contact point of a set of points P if, and only if, to each positive number ϵ and to each finite set of coördinates $\xi_{i_1}, \xi_{i_2}, \cdots, \xi_{i_n}$ there corresponds a point P of $[P]$ such that we have

$$|\xi_{i_s}(P) - \xi_{i_s}(P_0)| < \epsilon \qquad (s = 1, 2, \cdots, n).$$

Then, according to a theorem of Tychonoff,[14] every closed, bounded[15] subspace S of R is a bicompact, topological space (and, therefore, also, a normal space). Moreover, every bicompact, topological space can obviously be pictured as a closed, bounded subspace S of a space R. (We merely take as the coördinates $\xi_i(P)$ of a variable point P the set of all continuous, bounded functions of P.)

[14] A. Tychonoff, *Über die topologische Erweiterung von Räumen*, Math. Ann. Bd. 102 (1930).

[15] I.e., bounded in each coördinate.

Now suppose we wish to evaluate the closed connectivity ring of the space S. Then we may choose as a fundamental grating the grating Γ_p^0 cut out by the hyperplanes of the form $\xi_i = \lambda_{ij}$ (λ_{ij} rational), i.e., the grating consisting of all indexed covering pairs (a_{ij}, c_{ij}) such that a_{ij} and c_{ij} are determined by the conditions $\xi_i \leqq \lambda_{ij}$ and $\xi_i \geqq \lambda_{ij}$ respectively. For, given two closed subspaces α_p and γ_p of S with the intersection 0, it follows at once from the normality of S that there exists a finite approximation Γ_p^f of Γ_p^0 such that no 0-cell of Γ_p^f meets both α_p and γ_p. If the dimensionality of the containing space R is countable, the grating Γ_p^0 will be composed of a countable number of covering pairs.

In connection with the above example, we have one further remark to make which is often of considerable practical use. Suppose it is possible to find a finite grating Γ_p^* made up of closed covering pairs of the space S, such that the 0-cells (and therefore all the cells) of Γ_p^* are convex. Then the connectivity ring $\Xi(\Gamma_p^*)$ of the grating Γ_p^* is completely isomorphic with the closed connectivity ring $\Xi(\Gamma_p)$ of S. Indeed, suppose we form a refinement Γ_p^γ of Γ_p^* by adjoining to Γ_p^* any one covering pair (a_p, c_p) of the fundamental grating Γ_p^0 determined by the hyperplanes $\xi_i = \lambda_{ij}$. Then, since the cells of Γ_p^* are convex, no cell F_i^* of Γ_p^* will meet both a_p and c_p without meeting the barrier b_p. Therefore, by [16:4], we shall have $\Xi(\Gamma_p^r) \equiv \Xi(\Gamma_p^*)$. Moreover, the cells of Γ_p^r will also be convex, since each will be of one of the forms $a_pF_i^*$, $b_pF_i^*$, or $c_pF_i^*$. Thus it follows, by induction, that if Γ_p^f is any finite approximation of Γ_p^0 we shall have $\Xi(\Gamma_p^f \smile \Gamma_p^*) \equiv \Xi(\Gamma_p^*)$; whence, by [16:3], $\Xi(\Gamma_p^0) \equiv \Xi(\Gamma_p^*)$.

22. Within the ring of the closed chains Π_p there is an important sub-system Ω_b consisting of all chains of Γ_p with bicompact loci.

THEOREM [22:1]. *Let Π_p be the ring of all the closed chains of a space x, and let Ω_b be the sub-system of Π_p consisting of the chains of Π_p that are bicompact (i.e. that have bicompact loci). Then Ω_b is a preferred ideal of Π_p.*

PROOF. If K and L are arbitrary chains of the system Ω_b, so also are the chains of $K \pm L$, by [14:2], [3:6], and [3:4]. If K is a chain of Ω_b and L an arbitrary chain of Π_p, the chains KL and LK both belong to Ω_b, by [14:3] and [3:4]. Therefore, Ω_b is an ideal of Π_p. If K is a chain of Ω_b, so also are its components, by [14:4] and [3:4], and so also is its boundary, by [14:5] and [3:4]. Therefore, Ω_b is a preferred ideal.

Thus we have three different systems, Π_p, Ω_b, and Ω_o, which can be combined in any of the ways indicated in §13 so as to determine invariants of the space x. Perhaps the most satisfactory generalization of the classical connectivity ring of a manifold is the ring $\Xi(\Omega_b \bmod \Omega_o)$, rather than the more obvious ring $\Xi(\Pi_p \bmod \Omega_o)$. The two rings are, of course, identical if the space x is bicompact. Suppose, however, the space x is a region in n-space or, more generally, an open manifold of the classical type. Then the analogues of the classical $(n - \tau)$-chains of x will be the closed τ-chains (as defined in this paper) such that their loci are bicompact. We shall call the ring $\Xi(\Omega_b \bmod \Omega_o)$ the *bicompact connectivity ring* of x.

23. We shall bring the discussion to a close by proving a very general form of the topological duality theorem. To begin with, let us consider an arbitrary normal bicompact space 1, and two complementary subspaces x and y of 1, such that x is closed and y open. Moreover, let us ask ourselves to what extent the invariants of the subspaces x and y are determined by the covering pairs of the containing space 1.

First, we notice that the intersection with y of each closed covering pair (a_p, c_p) of 1 is a closed covering pair (a_p^y, c_p^y) of y. Furthermore, each closed covering pair of y may be thought of as the intersection (a_p^y, c_p^y) with y of the closed covering pair $(a_p^y \smile x, c_p^y \smile x)$ of 1. Therefore, if Γ_p is the grating of the closed covering pairs of 1 and Γ_p^y the intersection of Γ_p with y, the grating Γ_p^y must be made up of replicas of all the covering pairs of y; whence, by [16:6], its connectivity ring $\Xi(\Pi_p^y \bmod \Omega_o^y)$ must be completely isomorphic with the closed connectivity ring of y. We can evaluate this last ring in still another way. Since the spaces x and y are complementary, the ideal Ω_x^y formed by the chains of Π_p^y with loci dominated by x will be identical with the ideal Ω_o^y formed by the chains with loci dominated by 0. Hence, by applying Relation (15:1) with the rôles of x and y interchanged, we shall have

$$\Xi(\Pi_p^y \bmod \Omega_o^y) \equiv \Xi(\Pi_p \bmod \Omega_x). \tag{23:1}$$

The situation with reference to the intersection Γ_p^x of the grating Γ_p with x is a trifle more complicated. The intersection with x of a closed covering pair (a_p, c_p) of 1 is, indeed, a closed covering pair (a_p^x, c_p^x) of x, but, in this case, we cannot assert that every closed covering pair of x is an intersection of the form (a_p^x, c_p^x). It is easy to see, however, that the covering pairs of the form (a_p^x, c_p^x) constitute a fundamental set. Indeed, every open covering pair of x may be thought of as the intersection (a_q^x, c_q^x) with x of an open covering pair $(a_q^x \smile y, c_q^x \smile y)$ of 1. Moreover, since the space 1 is normal, the covering pair $(a_q^x \smile y, c_q^x \smile y)$ dominates a closed covering pair (a_p, c_p) of 1 intersecting x in a closed covering pair of x of the form (a_p^x, c_p^x). Hence, by §21, the grating Γ_p^x is made up of a fundamental set of closed covering pairs of x. Hence, also, the closed connectivity ring of x is expressible in either of the forms

$$\Xi(\Pi_p^x \bmod \Omega_o^x) \equiv \Xi(\Pi_p \bmod \Omega_y). \tag{23:2}$$

One Further Remark. If the locus of a closed chain K_p of 1 is dominated by y, the intersection K_p^y of K_p with y is necessarily a bicompact chain of y, since its locus coincides with the locus of K_p and is, therefore, a closed subspace of the bicompact space 1. Let us assume that *every bicompact subspace of* 1 *is closed with respect to* 1.[16] Then we can further assert that every bicompact chain of y is congruent, mod Ω_o^y, to the intersection K_p^y with y of a closed chain K_p of 1 dominated by y. For a bicompact chain of y is a closed chain of y and may, therefore, be thought of as the intersection L_p^y with y of *some* closed

[16] This property is apparently not implied by the assumptions of §3.

chain L_p of 1, where L_p is not necessarily dominated by y (cf. the derivation of (23:1) above). Moreover, by the special assumption about the bicompact subspaces of 1, the locus $| L_p^y |$ is closed with respect to 1. Therefore, since the space 1 is normal, and since we have

$$x \frown | L_p^y | = 0,$$

it follows from [20:3] that there exists a closed covering pair (a_p, c_p) of 1 such that we have

$$(23:3) \qquad a_p \frown x = 0, \qquad c_p \frown | L_p^y | = 0.$$

Thus, we may write $K_p = a_p L_p$. By the first relation in (23:3), the chain K_p will be dominated by y, since it fails to meet x. Moreover, by the second relation in (23:3), we shall have $| c_p^y L_p^y | = 0$; whence, $K_p^y = a_p^y L_p^y \equiv (a_p^y + c_p^y) L_p^y = L_p^y$, which is what we set out to prove.

We shall say that the space 1 is *simply connected* if its closed connectivity ring $\Xi(\Pi_p \bmod \Omega_o)$ consists only of the elements determined by the various multiples of the 0-cycle 1 and is, therefore, isomorphic with the ring of all the integers.

THEOREM [23:1]. *Let* 1 *be any simply connected, normal, bicompact space with the property that every bicompact subspace of* 1 *is closed. Moreover, let* x *and* y *be two complementary proper subspaces of* 1 *such that* x *is closed and* y *open. Then the* τ^{th} *bicompact connectivity group of* x *is simply isomorphic with the* $(\tau + 1)^{\text{st}}$ *bicompact connectivity group of* y.

Throughout the proof it is to be understood that we are operating modulo the empty chains. We form the grating Γ_p of all the closed covering pairs of the containing space 1 and evaluate the bicompact connectivity groups of x and y in terms of the bicompact chains of the intersections Γ_p^x and Γ_p^y of Γ_p with x and y respectively. Let K be a τ-chain of Γ_p with boundary K' dominated by y. Thus K^x is a bicompact τ-cycle of Γ_p^x and $(K')^y$ is a bicompact $(\tau + 1)$-cycle of Γ_p^y. Moreover, the correspondence $K^x : (K')^y$ thus induced by the τ-chains K has the property that every bicompact τ-cycle of Γ_p^x and every bicompact $(\tau + 1)$-cycle of Γ_p^y appears in it at least once. Finally, by the argument used in proving [13:3] the cycle K^x bounds a bicompact chain of Γ_p^x if, and only if, the cycle $(K')^y$ bounds a bicompact chain of Γ_p^y. The isomorphism of the τ^{th} bicompact connectivity group of x and the $(\tau + 1)^{\text{st}}$ bicompact connectivity group of y is, therefore, established.

ILLUSTRATION. Let the space 1 be the hypercube

$$0 \leqq \xi_i \leqq 1$$

in the space of R of a (finite or transfinite) set of real variables ξ_i, and let x be any closed (and, therefore, bicompact) subspace of 1. Since the hypercube 1 is convex, it follows immediately from the discussion at the end of §21 that it is simply connected. Theorem [23:1] is, therefore, applicable to the subspace x and its complement y.

INSTITUTE FOR ADVANCED STUDY.

ANNALS OF MATHEMATICS
Vol. 39, No. 4, October, 1938

LINEAR FUNCTIONALS ON CERTAIN SPACES OF ABSTRACTLY-VALUED FUNCTIONS

BY S. BOCHNER AND A. E. TAYLOR[1]

(Received May 27, 1938)

1. Introduction

This paper is concerned with the study of certain classes of abstractly-valued functions which arise naturally in connection with the theory of integration of functions with values in a Banach space. These classes correspond to the usual classes C, L^p, M, etc. which are well known in analysis. The subject matter of the paper is divided into seven sections as follows: §2 contains most of the technical apparatus, which consists of the Lebesgue integral of abstract functions as defined by Bochner,[2] and several types of integrals based on the notion of the Riemann-Stieltjes integral. Since, in the case of abstractly-valued functions, there may be a distinction between an indefinite integral and an absolutely continuous function,[3] we have found it important to introduce the class of functions of finite p-variation, corresponding to the classical criterion for a function's being the indefinite integral of a function of class L^p. For certain Banach spaces it is known that the distinction mentioned above does not arise.[4] Such a space is said to satisfy condition (D) [for the precise statement see §2]. We have obtained some information about the implications of this condition and its relation to other properties of the space [see especially §§4, 5 and 7]. §3 contains the determination of the most general linear functionals on the various function classes under discussion. §4 is devoted to a discussion of weak convergence and its relation to condition (D). In §5 the interrelations of condition (D), weak completeness, and reflexiveness of Banach spaces is taken up. In §6 we have results on the Parseval relation and properties of the Fourier coefficients of abstract functions. Counter examples to show the divergence from classical theory are given. In §7 we return to the question of reflexivity and condition (D), obtaining a criterion for spaces which satisfy this condition. Finally, in §8 it is shown how by the methods of Fourier series it is possible to characterize

[1] National Research Fellow.

[2] S. Bochner, *Integration von Funktionen, deren Werte die Elemente eines Vektorraumes sind*, Fundamenta Mathematicae, vol. 20 (1933), pp. 262–276. We shall refer to this paper as Bochner I.

[3] S. Bochner, *Absolut-additive abstracte Mengenfunktionen*, Fundamenta Mathematicae, vol. 21 (1933), pp. 211–213. See also the paper of Clarkson referred to in footnote 4).

[4] J. A. Clarkson, *Uniformly Convex Spaces*, Transactions of the American Mathematical Society, vol. 40 (1936), pp. 396–414.

Nelson Dunford and A. P. Morse, *Remarks on the Preceding Paper of James A. Clarkson.* Transactions of the American Mathematical Society, vol. 40 (1936) pp. 415–420.

linear operations on an arbitrary Banach space X to one of the well known function spaces L^p, M, C.

2. Integration of Abstract Functions

A definition of the Lebesgue integral for functions on a real interval to a complex Banach space X has been given by Bochner.[5] Using Bochner's definition of measurability and summability we introduce the following classes of functions. We choose for convenience the interval $(0, 2\pi)$ and assume all functions to be defined almost everywhere (a.e.) and measurable. $L^p\{X\}(p \geqq 1)$: the class of functions f with values in X for which $||f|| = \left(\int_0^{2\pi} ||f(t)||^p\, dt\right)^{1/p}$ exists. With this definition of norm $L^p\{X\}$ is a Banach space.[6] If $p = 1$ we omit the index. $M\{X\}$: the class of essentially bounded functions f with values in X. [f is said to be essentially bounded if there exists a number $A \geqq 0$ such that the set where $||f(t)|| > A$ is of measure zero. The least such constant A is called $\sup^{\circ}_{0 \leqq t \leqq 2\pi} ||f(t)||$]. With $||f|| = \sup^{\circ}_{0 \leqq t \leqq 2\pi} ||f(t)||$ $M\{X\}$ is a Banach space. As a class it is contained in all the classes $L^p\{X\}$.

If X is the complex number system we omit the X from the above symbols.

We shall also be interested in the following classes of functions:

$V^p\{X\}$ $(p \geqq 1)$: the class of functions f with values in X, defined everywhere on $(0, 2\pi)$ and such that the sums

$$\sum_{\nu=1}^{n} \frac{||f(t_\nu) - f(t_{\nu-1})||^p}{|t_\nu - t_{\nu-1}|^{p-1}}$$

are bounded for all partitions $\{t_\nu\}$, $0 = t_0 < t_1 < \cdots < t_n = 2\pi$. The least upper bound of all such sums is denoted by $V^p(f)$. We define $||f|| = ||f(0)|| + \{V^p(f)\}^{1/p}$. With this norm $V^p\{X\}$ is a Banach space. (The completeness of $V^p\{X\}$ follows from the easily proved semi-continuity property: if $f_n, f \in V^p\{X\}$ and if $f_n(t) \to f(t)$ for $0 \leqq t \leqq 2\pi$ then $\{V^p(f)\}^{1/p} \leqq \lim_{n\to\infty} \{V^p(f_n)\}^{1/p}$.

If $p = 1$ we write $V\{X\}$ instead of $V^1\{X\}$; functions of this class are said to be of bounded variation.

$AC\{X\}$: the subset of $V\{X\}$ for which the sum $\sum_\nu ||f(t_\nu + h_\nu) - f(t_\nu)||$, taken over any set of non-overlapping intervals $(t_\nu, t_\nu + h_\nu)$, tends to zero with $\sum_\nu h_\nu$. $AC\{X\}$ may be regarded as a complete subspace of $V\{X\}$.

$V^\infty\{X\}$: the class of functions f defined everywhere on $(0, 2\pi)$, with values in X, and such that for some constant $A \geqq 0$ $||f(t + h) - f(t)|| \leqq A|h|$ whenever $t, t + h$ are on $(0, 2\pi)$. The least such constant A is denoted by $N(f)$; we write $||f|| = ||f(0)|| + N(f)$. With this norm $V^\infty\{X\}$ is a Banach space. As a class it is contained in all the classes $V^p\{X\}$.

---

[5] See Bochner I. The present paper presupposes a knowledge of Bochner's definition of the integral.

[6] For facts and terminology concerning Banach spaces we shall refer to Banach, Opérations Linéaires, Warsaw, 1932. We shall deal throughout with *complex* spaces.

All the classes $V^p\{X\}$ with $1 < p \leqq \infty$ are contained in $AC\{X\}$. The indefinite integral of a function of class $L^p\{X\}$ $(p > 1)$ is in $V^p\{X\}$; the indefinite integral of a function of class $L\{X\}$ is in $AC\{X\}$; the indefinite integral of a function of class $M\{X\}$ is in $V^\infty\{X\}$.[7]

It is well known that for some spaces X a function f may be in $V\{X\}$, or even in $AC\{X\}$ or $V^\infty\{X\}$, without being differentiable at any point. For a number of the usual spaces of analysis, for instance if $X = L^p$ $(p > 1)$ or if $X = l^p$ $(p \geqq 1)$, this situation cannot occur, and the relation between differentiation and integration is as in the ordinary case of numerical functions (see footnotes (3), (4)). Accordingly we lay down the following definition:

DEFINITION *A Banach space X will be said to have the property* (D) *if whenever $f \in V\{X\}$ then f has a derivative* a.e.

If X has property (D) the following things are true:

(I) A function $f \in AC\{X\}$ is the indefinite integral of its derivative f', and $V(f) = \int_0^{2\pi} \|f'(t)\|\, dt$.

(II) If $f \in V^p\{X\}$ $(p > 1)$ then $f' \in L^p\{X\}$ and $V^p(f) = \int_0^{2\pi} \|f'(t)\|^p\, dt$.

(III) If $f \in V^\infty\{X\}$ then $f' \in M\{X\}$ and $N(f) = \sup^\circ \|f'(t)\|$.

REMARK: If we denote by $AC_0\{X\}$, $V_0^p\{X\}$, $V_0^\infty\{X\}$ the subclasses of $AC\{X\}$, $V^p\{X\}$, $V^\infty\{X\}$, respectively, for which the elements f have the property $f(0) = 0$ it is clear from the above that when X has property (D) the spaces $AC_0\{X\}$, $V_0^p\{X\}$, $V_0^\infty\{X\}$ are equivalent,[8] respectively, to the spaces $L\{X\}$, $L^p\{X\}$, $M\{X\}$.

The proofs of I–III are not essentially different from the corresponding proofs for numerical functions, and we shall omit them.

We shall introduce several types of integrals of abstract functions; the basic idea behind each type is the Riemann-Stieltjes integral. We begin with the following considerations. If X is a complex Banach space denote by $C\{X\}$ the class of functions f defined and continuous on the interval $(0, 2\pi)$, with values in X. With $\|f\| = \max_{0 \leqq t \leqq 2\pi} \|f(t)\|$ $C\{X\}$ is a Banach space, and in fact a closed subspace of $M\{X\}$. If Y is a second Banach space we denote by $[X, Y]$ the class of all linear operations T on X to Y. With $\|T\| = \sup_{\|x\| \leqq 1} \|Tx\|$ $[X, Y]$ is a Banach space. If Y is the complex number system $[X, Y]$ is denoted by X^* and T is called a linear functional. We shall refer to X^* as the space conjugate to X.

We are now ready to define the integral

$$\int_0^{2\pi} dT(t) f(t), \tag{2.1}$$

[7] The proofs of these facts do not differ from the corresponding proofs for numerical functions. See for instance, Titchmarsh, *Theory of Functions*, Oxford 1932, chapters XI, XII.

[8] For the concept of equivalence of two Banach spaces see Banach, loc. cit. p. 180.

where $T(t) \in V\{[X, Y]\}$ and $f(t) \in C\{X\}$. This is done as follows. Corresponding to a partition D: $0 = t_0 < t_1 < \cdots < t_n = 2\pi$ and points τ_ν, $t_{\nu-1} \leqq \tau_\nu \leqq t_\nu$, form the sum (an element of Y)

$$S(D) = \sum_{\nu=1}^{n} [T(t_\nu) - T(t_{\nu-1})]f(\tau_\nu).$$

If we put $|D| = \max_\nu |t_\nu - t_{\nu-1}|$ then it may be proved in the usual way that for a sequence of partitions D_i with $|D_i| \to 0$ the sums $S(D_i)$ approach a limit (in Y) which is independent of the particular sequence $\{D_i\}$. This limit is called the integral of f with respect to T, and written in the form (2.1). From the manner of definition it is clear that

$$\left\| \int_0^{2\pi} dT(t) f(t) \right\| \leqq \max \|f(t)\| \, V(T). \tag{2.2}$$

We shall make use of this integral in two special cases only—first when T is a functional, so that $[X, Y] = X^*$, and again when $Y = X$ and $T(t)x$ has the form $\alpha(t)x$, where $\alpha(t)$ is a *numerical* function of bounded variation.

In an exactly similar way we define the integral

$$\int_0^{2\pi} T(t)\, dF(t), \tag{2.3}$$

where $T(t) \in C\{[X, Y]\}$ and $F(t) \in V\{X\}$. The approximating sums have the form

$$S(D) = \sum_{\nu=1}^{n} T(\tau_\nu)[F(t_\nu) - F(t_{\nu-1})].$$

The integral is an element of Y satisfying the inequality

$$\left\| \int_0^{2\pi} T(t)\, dF(t) \right\| \leqq \max_t \|T(t)\| \, V(F). \tag{2.4}$$

We may specialize $T(t)$ in the same manner as above.

If in (2.1) $T(t)$ happens to be the indefinite (Lebesgue) integral of a function in $L\{[X, Y]\}$, say $T(t) = \int_0^t A(s)\, ds$, then $T'(t) = A(t)$ a.e. and the integral (2.1) may be expressed as a Lebesgue integral

$$\int_0^{2\pi} dT(t) f(t) = \int_0^{2\pi} A(t) f(t)\, dt. \tag{2.5}$$

Here $A(t)\, f(t)$ is a function with values in Y, integrable in the sense of Bochner. (2.5) may be proved by first showing that it is true if $A(t)$ is a step-function and then treating the general case by approximation in the mean.[9] We omit details. Similar remarks apply to the integral (2.3).

[9] For the possibility of approximating, in the mean, to any integrable function by step functions, see Bochner I.

We next propose to define an integral similar to (2.1) under somewhat different circumstances. We shall limit ourselves to the case that Y is the complex number system. Suppose, then, that $f(t) \in L^p\{X\}$ and that $\Phi(t) \in V^q\{X^*\}$ where $p > 1$ and $p^{-1} + q^{-1} = 1$ (we shall hereafter assume that when p and q are mentioned in the same connection they stand in this relation). We define an integral

$$\int_0^{2\pi} d\Phi(t) f(t) \tag{2.6}$$

as follows. If $f(t) \in C\{X\}$ it is defined as in (2.1); this integral will exist, since $V^q\{X^*\}$ is contained in $V\{X^*\}$. Now we see from the approximating sums that

$$|S(D)| \leqq \left(\sum_{\nu=1}^{n} \frac{||\Phi(t_\nu) - \Phi(t_{\nu-1})||^q}{|t_\nu - t_{\nu-1}|^{q-1}}\right)^{1/q} \left(\sum_{\nu=1}^{n} ||f(\tau_\nu)||^p \, |t_\nu - t_{\nu-1}|\right)^{1/p}$$

and hence that, for continuous $f(t)$,

$$\left|\int_0^{2\pi} d\Phi(t) f(t)\right| \leqq [V^q(\Phi)]^{1/q} \left(\int_0^{2\pi} ||f(t)||^p \, dt\right)^{1/p}. \tag{2.7}$$

Now the linear manifold of continuous functions is everywhere dense in $L^p\{X\}$ and on this manifold the integral, as defined, is a linear functional of norm $[V^q(\Phi)]^{1/q}$. It therefore admits a *unique* extension to define a linear function on $L^p\{X\}$, with the same norm.[10] We *define* the integral (2.6) in general by this extension. Then (2.7) is still valid.

In certain situations the integral (2.6) may be expressed as a Lebesgue integral. The precise theorem is the following

THEOREM 2.1. *If $\Phi(t)$ is an indefinite integral, say $\Phi(t) = \int_0^t \varphi(t)\, dt$ + constant, where $\varphi(t) \in L\{X^*\}$, then $\varphi \in L^q\{X^*\}$, $\varphi(t) f(t) \in L$, and*

$$\int_0^{2\pi} d\Phi(t) f(t) = \int_0^{2\pi} \varphi(t) f(t)\, dt. \tag{2.8}$$

PROOF: $\Phi'(t)$ exists and is equal to $\varphi(t)$ a.e. (see Bochner I). That $\varphi(t) \in L^q\{X^*\}$ is proved as in the classical case. It also follows that $V^q\{\Phi\} = \int_0^{2\pi} ||\varphi(t)||^q \, dt$. The truth of (2.8) for continuous $f(t)$ is a consequence of (2.5). For the general case we need to know that $\varphi(t) f(t) \in L$ and that

$$\left|\int_0^{2\varphi} \varphi(t) f(t)\, dt\right| \leqq \left(\int_0^{2\pi} ||\varphi(t)||^q \, dt\right)^{1/q} \left(\int_0^{2\pi} ||f(t)||^p \, dt\right)^{1/p}. \tag{2.9}$$

[10] If a functional U is defined and linear on a linear manifold X_0 which is everywhere dense in X, with $|U(x)| \leqq A\,||x||$ for all $x \in X_0$ then U admits a unique linear extension to the whole of X in such a way that the above inequality continues to hold.

The fact that the continuous functions (and likewise the step functions) are everywhere dense in $L^p\{X\}$ is used at a number of places later in the paper.

These facts are proved by appealing directly to the definition of measurability (Bochner I) and then proceeding as in the usual proof of Holder's inequality. Finally, the truth of (2.8) is deduced by approximating $f(t)$ in mean by continuous functions $f_n(t)$. If $\int_0^{2\pi} \|f(t) - f_n(t)\|^p\, dt \to 0$ the left side of (2.8) is *defined* as the limit of $\int_0^{2\pi} d\Phi(t) f_n(t) = \int_0^{2\pi} \varphi(t) f_n(t)\, dt$. On the other hand this last expression converges to the right member of (2.8), as follows from (2.9).

An entirely analogous discussion may be carried out for an integral similar to (2.6) where it is assumed that $\Phi(t) \in V^\infty\{X\}$ and $f(t) \in L\{X\}$. For continuous $f(t)$ the integral is defined as in (2.1). From the approximating sums it follows that in this case

$$\left| \int_0^{2\pi} d\Phi(t) f(t) \right| \leqq N(\Phi) \int_0^{2\pi} \|f(t)\|\, dt. \tag{2.10}$$

The integral is thus a linear functional defined on the linear manifold of continuous functions in $L\{X\}$; since this manifold is everywhere dense the functional has a unique linear extension to all of $L\{X\}$. This extension defines the integral in general. (2.10) remains valid. Corresponding to Theorem 2.1 we state

THEOREM 2.2. *If $\Phi(t) \in V^\infty\{X^*\}$ is an indefinite integral, say $\Phi(t) = \int_0^t \varphi(t)dt$ + constant, then $\varphi(t) \in M\{X^*\}$, $\varphi(t) f(t) \in L$ if $f(t) \in L\{X\}$, and*

$$\int_0^{2\pi} d\Phi(t) f(t) = \int_0^{2\pi} \varphi(t) f(t)\, dt. \tag{2.11}$$

We omit the proof, merely noting that (2.10) becomes, in this case

$$\left| \int_0^{2\pi} \varphi(t) f(t)\, dt \right| \leqq \sup_t{}^{\circ} \|\varphi(t)\| \int_0^{2\pi} \|f(t)\|\, dt. \tag{2.12}$$

Finally, we shall state explicitly for later use the two following properties of the Lebesgue integral: (1) If $f(t) \in L\{X\}$ and $\varphi \in X^*$ then $\varphi f(t) \in L$ and

$$\int_0^{2\pi} \varphi f(t)\, dt = \varphi\left(\int_0^{2\pi} f(t)\, dt \right); \tag{2.13}$$

(2) if $\varphi(t) \in L\{X^*\}$ and $x \in X$ then $\varphi(t)x \in L$ and

$$\int_0^{2\pi} \varphi(t) x\, dt = \psi(x), \quad \text{where} \quad \psi = \int_0^{2\pi} \varphi(t)\, dt. \tag{2.14}$$

These are provable directly from Bochner's definition of the integral.

3. The Form of Linear Functionals

We begin with the space $C\{X\}$.

THEOREM 3.1. *If U is a linear functional defined on $C\{X\}$ there exists a function $\Phi(t) \in V\{X^*\}$ such that $V(\Phi) = \|U\|$ and for each $f(t) \in C\{X\}$*

$$U(f) = \int_0^{2\pi} d\Phi(t) f(t), \tag{3.1}$$

the integral on the right being a Riemann-Stieltjes integral of type (2.1). *For any $\Phi(t) \in V\{X^*\}$ the above integral defines a linear functional of norm $\leqq V(\Phi)$.*

PROOF: Since $C\{X\}$ is a subspace of $M\{X\}$ we may, by the Hahn-Banach theorem,[11] extend U to the whole of $M\{X\}$ without increasing its norm. We shall use the same symbol to denote the extended functional. Corresponding to an $x \in X$ and a real number s, $0 \leqq s \leqq 2\pi$ define an element $f_{s,x} \in M\{X\}$ as follows

$$f_{0,x}(t) = 0,$$

$$f_{s,x}(t) = \begin{cases} x, & 0 \leqq t \leqq s, \\ 0, & s < t \leqq 2\pi, \end{cases} \qquad s > 0. \tag{3.2}$$

Then $U(f_{s,x})$ defines, for each s, a linear functional on X. We write

$$\Phi(s)x = U(f_{s,x}) \tag{3.3}$$

and observe that $\Phi(0) = 0$, $\|\Phi(s)\| \leqq \|U\|$. $\Phi(s)$ is an element of $V\{X^*\}$. For if $x_1, \cdots, x_n \in X$ and $0 = t_0 < t_1 < \cdots < t_n = 2\pi$ is a partition

$$\sum_{\nu=1}^{n} |\Phi(t_\nu)x_\nu - \Phi(t_{\nu-1})x_\nu| = \sum_{\nu=1}^{n} |U(f_{t_\nu,x_\nu}) - U(f_{t_{\nu-1},x_\nu})|$$

$$= U\left(\sum_{\nu=1}^{n} \epsilon_\nu [f_{t_\nu,x_\nu} - f_{t_{\nu-1},x_\nu}]\right) \leqq \|U\| \max_\nu \|x_\nu\|,$$

where the ϵ_ν are appropriately chosen complex numbers of unit modulus. Hence, if $\epsilon > 0$ is given, and we choose x_ν so that $\|x_\nu\| = 1$ and

$$|\Phi(t_\nu)x_\nu - \Phi(t_{\nu-1})x_\nu| > \|\Phi(t_\nu) - \Phi(t_{\nu-1})\| - \frac{1}{n}\epsilon$$

it follows that

$$\sum_{\nu=1}^{n} \|\Phi(t_\nu) - \Phi(t_{\nu-1})\| \leqq \|U\| + \epsilon$$

from which we infer $V(\Phi) \leqq \|U\|$. It then follows, in the usual way[12] that $U(f)$ has the form (3.1). The last assertion of the theorem is clear from §2.

---

[11] Banach, loc. cit. p. 55, Theorem 2. For complex spaces the theorem was recently proved by H. F. Bohnenblust and A. Sobczyk, Bulletin of the American Mathematical Society, vol. 44 (1938), pp. 91–93.

[12] Banach, loc. cit. p. 60. To shorten the writing we have whenever possible omitted arguments which do not differ essentially from those used in Banach's book.

Next we consider $L^p\{X\}$, $p > 1$.

THEOREM 3.2. *To a linear functional U defined on $L^p\{X\}$ there corresponds a function $\Phi(t) \in V^q\{X^*\}$ such that $V^q(\Phi) = ||U||^q$, and U is given by the formula* (3.1) *where the integral is of type* (2.6). *Conversely, for any $\Phi(t) \in V^q\{X^*\}$ the integral defines a linear functional of norm $\leqq [V^q(\Phi)]^{1/q}$. In case the space X^* satisfies condition* (D) *the functional takes the form.*

$$U(f) = \int_0^{2\pi} \varphi(t) f(t)\, dt, \tag{3.4}$$

where $\varphi(t) \in L^q\{X^*\}$ and $||U||^q = \int_0^{2\pi} ||\varphi(t)||^q\, dt$.

PROOF: Define $f_{s,x}(t)$ and $\Phi(s)$ by (3.2), (3.3). For $x_1, \cdots, x_n \in X$ and a partition $\{t_\nu\}$, proceeding as we did above we find that

$$\sum_{\nu=1}^{n} |\Phi(t_\nu)x_\nu - \Phi(t_{\nu-1})x_\nu| \leqq ||U|| \left(\sum_{\nu=1}^{n} ||x_\nu||^p\, |t_\nu - t_{\nu-1}|\right)^{1/p}.$$

Now for any positive ϵ and non-negative numbers $b_1, \cdots, b_n$ we can choose x_ν, $||x_\nu|| = b_\nu$, so that

$$|\Phi(t_\nu)x_\nu - \Phi(t_{\nu-1})x_\nu| > ||\Phi(t_\nu) - \Phi(t_{\nu-1})||\, ||x_\nu|| - \frac{\epsilon}{n}.$$

Then

$$\sum_{\nu=1}^{n} ||\Phi(t_\nu) - \Phi(t_{\nu-1})||\, b_\nu < ||U|| \left(\sum_{\nu=1}^{n} b_\nu^p\, |t_\nu - t_{\nu-1}|\right)^{1/p} + \epsilon,$$

and so

$$\sum_{\nu=1}^{n} ||\Phi(t_\nu) - \Phi(t_{\nu-1})||\, b_\nu \leqq ||U|| \left(\sum_{\nu=1}^{n} b_\nu^p\, |t_\nu - t_{\nu-1}|\right)^{1/p}.$$

If we put $a_\nu = ||\Phi(t_\nu) - \Phi(t_{\nu-1})||$, $b_\nu = [a_\nu/(t_\nu - t_{\nu-1})]^{q/p}$, then the above inequality becomes

$$\sum_{\nu=1}^{n} \frac{||\Phi(t_\nu) - \Phi(t_{\nu-1})||^q}{|t_\nu - t_{\nu-1}|^{q-1}} \leqq ||U||^q,$$

and so $\Phi(t) \in V^q\{X^*\}$ with $V^q(\Phi) \leqq ||U||^q$. On the other hand, if $f_n(t)$ is a step function which takes on the value x_ν in the interval $t_{\nu-1} < t \leqq t_\nu$, then

$$f_n(t) = \sum_{\nu=1}^{n} [f_{t_\nu, x_\nu}(t) - f_{t_{\nu-1}, x_\nu}(t), \tag{3.5}$$

$$\int_0^{2\pi} ||f_n(t)||^p\, dt = \sum_{\nu=1}^{n} ||x_\nu||^p\, |t_\nu - t_{\nu-1}|,$$

and

$$|U(f_n)| = \left|\sum_{\nu=1}^{n} [\Phi(t_\nu)x_\nu - \Phi(t_{\nu-1})x_\nu]\right| \leqq \sum_{\nu=1}^{n} ||\Phi(t_\nu) - \Phi(t_{\nu-1})||\, ||x_\nu||$$

$$\leqq \left(\sum_{\nu=1}^{n} \frac{||\Phi(t_\nu) - \Phi(t_{\nu-1})||^q}{|t_\nu - t_{\nu-1}|^{q-1}}\right)^{1/q} \left(\sum_{\nu=1}^{n} ||x_\nu||^p\, |t_\nu - t_{\nu-1}|\right)^{1/p},$$

or

$$|U(f_n)| \leqq [V^q(\Phi)]^{1/q}\left(\int_0^{2\pi} ||f_n(t)||^p\,dt\right)^{1/p},$$

whence if $f \in L^p\{X\}$, and f_n is chosen so that $\int_0^{2\pi} ||f(t) - f_n(t)||^p dt \to 0$, we see that $||U|| \leqq [V^q(\Phi)]^{1/q}$. Therefore $||U|| = [V^q(\Phi)]^{1/q}$.

If $f(t)$ happens to be continuous it is easily seen that $U(f)$ is given by (3.1), the integral being of type (2.1). The rest of the theorem follows from the results of §2.

Finally, consider $L\{X\}$.

THEOREM 3.3. *To a linear functional U defined on $L\{X\}$ there corresponds a function $\Phi(t) \in V^\infty\{X^*\}$ such that $N(\Phi) = ||U||$, and U is given by* (3.1) *where the integral is of the type meant in* (2.10). *Conversely, for any $\Phi(t) \in V^\infty\{X^*\}$ the integral defines a linear functional of norm $< N(\Phi)$. If X^* satisfies condition* (D) *the functional takes the form* (3.4) *where $\varphi(t) \in M\{X^*\}$ and $||U|| = \sup^{\circ}_{0\leqq t\leqq 2\pi} ||\varphi(t)||$.*

PROOF: Defining $f_{s,x}$, $\Phi(s)$ as in the previous cases we see that if s and $s+h$ are in $(0, 2\pi)$, $(h > 0)$, then

$$|\Phi(s+h)x - \Phi(s)x| = |U(f_{s+h,x}) - U(f_{s,x})| \leqq ||U|| \int_s^{s+h} ||x||\,dt.$$

Hence $||\Phi(s+h) - \Phi(s)|| \leqq ||U||\,|h|$, so that $\Phi(s) \in V^\infty\{X^*\}$ and $N(\Phi) \leqq ||U||$. But for the step function $f_n(t)$ of (3.5)

$$|U(f_n)| \leqq \sum_{\nu=1}^{n} ||\Phi(t_\nu) - \Phi(t_{\nu-1})||\,||x_\nu|| \leqq N(\Phi) \sum_{\nu=1}^{n} |t_\nu - t_{\nu-1}|\,||x_\nu||,$$

$$|U(f_n)| \leqq N(\Phi) \int_0^{2\pi} ||f_n(t)||\,dt,$$

whence it follows that for *any* $f(t) \in L\{X\}$, $|U(f)| \leqq N(\Phi)\,||f||$, that is, $||U|| \leqq N(\Phi)$. Hence $||U|| = N(\Phi)$. The rest of the theorem follows as in the treatment of $L^p\{X\}$, by referring to the discussion in §2.

We conclude this section with a few remarks about conjugate spaces. The space X^* of all linear functionals on X is called the conjugate space of X. From the results just obtained it is easy to verify that the conjugate spaces of $L^p\{X\}$ $(p > 1)$, $L\{X\}$ are equivalent,[8] respectively, to the spaces $V_0^q\{X^*\}$, $V_0^\infty\{X^*\}$ defined in §2. For instance, in the case of $L^p\{X\}$ we have shown that to a linear functional U corresponds a function $\Phi(t) \in V_0^q\{X\}^*$ with $||U||^q = V^q(\Phi)$. This correspondence is clearly linear and isometric. To demonstrate the equivalence of $[L^p\{X\}]^*$ and $V_0^q\{X^*\}$ it is merely necessary to show that if $\Phi(t) \in V_0^q\{X^*\}$ then for the functional U defined by (3.1) we have $U(f_{s,x}) = \Phi(s)x$. This is clearly true for $s = 0$. If $s > 0$ define, for $x \in X$,

$$f_n(t) = \begin{cases} x, & 0 \leqq t \leqq s \\ -n\left(t - s - \dfrac{1}{n}\right), & s \leqq t \leqq s + \dfrac{1}{n}, \\ 0, & s + \dfrac{1}{n} \leqq t \leqq 2\pi. \end{cases}$$

Then $f_n(t)$ is continuous and $f_n \to f_{s,x}$ in the metric of $L^p\{X\}$. Hence $U(f_n) \to U(f_{s,x})$. But

$$U(f_n) = \int_0^{2\pi} d\Phi(t) f_n(t) = \Phi(s)x + \int_s^{s+1/n} d\Phi(t)\left[n\left(s + \frac{1}{n} - t\right)x\right],$$

$$|U(f_n) - \Phi(s)x| \leqq ||x|| \cdot \left\{\text{variation of } \Phi \text{ over } \left(s, s + \frac{1}{n}\right)\right\},$$

and the right side approaches zero with $1/n$. Hence $U(f_{s,x}) = \Phi(s)x$ as we wished to prove.

Similar remarks apply to the conjugate space of $L\{X\}$.

4. Weak Convergence

In this section we shall consider the weak convergence[13] of sequences in the spaces $L^p\{X\}$, $p \geqq 1$, and $C\{X\}$. In the latter case it is shown that the criterion for weak convergence is a direct generalization of the criterion for the ordinary space of continuous functions C. A similar generalization is valid for the spaces $L^p\{X\}$ *if* X^* *satisfies condition* (D), *but not necessarily otherwise*, as we demonstrate by a counter-example.

Consider first $L^p\{X\}$, $p > 1$.

THEOREM 4.1. *If* X^* *satisfies the condition* (D) *then in order that a sequence* $\{f_n\}$ *of elements of* $L^p\{X\}$ *converge weakly to an element* $f \in L^p\{X\}$ *it is both necessary and sufficient that*

1) *the sequence* $\int_0^{2\pi} ||f_n(t)||^p\, dt$ *be bounded, and*

2) $\lim_{n\to\infty} \varphi\left(\int_0^s f_n(t)\, dt\right) = \varphi\left(\int_0^s f(t)\, dt\right)$ *for each* s, $0 \leqq s \leqq 2\pi$, *and each* $\varphi \in X^*$.

PROOF: The conditions are necessary, for it follows from the weak convergence that the sequence of norms $||f_n||$ is bounded,[14] and $\varphi\left(\int_0^s f(t)\, dt\right)$ is a particular functional on $L^p\{X\}$.

The conditions are also sufficient. For since X^* satisfies condition (D), a linear functional U has the form (3.4). Now if $\varphi(t)$ is a step-function, say $\varphi(t) = \varphi_\nu$ in the interval $(t_{\nu-1} < t \leqq t_\nu)$, where $\{t_\nu)\}$ is a partition, and $\varphi_1, \cdots, \varphi_n$ are in X^*, then

$$\int_0^{2\pi} \varphi(t) f(t)\, dt = \sum_{\nu=1}^{n} \int_{t_{\nu-1}}^{t_\nu} \varphi_\nu f(t)\, dt = \sum_{\nu=1}^{n} \varphi_\nu\left(\int_{t_{\nu-1}}^{t_\nu} f(t)\, dt\right)$$

where we make use of (2.13). It follows easily from this and condition 2) that

$$\int_0^{2\pi} \varphi(t) f_n(t)\, dt \to \int_0^{2\pi} \varphi(t) f(t)\, dt$$

---

[13] Banach, loc. cit. p. 133. A sequence $\{x_n\}$ of elements of X is said to converge weakly to $x \in X$ if $\varphi(x_n) \to \varphi(x)$ for each linear functional φ defined on X (i.e. $\varphi \in X^*$).

[14] Banach, loc. cit. p. 133, Theorem 1.

if $\varphi(t)$ is a step function. Now the step functions form an everywhere dense set in $L^q\{X^*\}$, which is in this case, since X^* satisfies condition (D), equivalent to the conjugate space of $L^p\{X\}$. Therefore, by the usual reasoning[15] it follows that f_n converges weakly to f.

We next consider $L\{X\}$.

THEOREM 4.2. *If X^* satisfies condition* (D) *then in order that a sequence $\{f_n\}$ of elements of $L\{X\}$ converge weakly to an element $f \in L\{X\}$ it is both necessary and sufficient that*

(1) *Given $\epsilon > 0$, there exist a $\delta > 0$ such that $\int_S ||f_n(t)||\,dt \leqq \epsilon$ for all sets S on $(0, 2\pi)$ of measure less than δ,*

(2) $\lim_{n\to\infty} \varphi\left(\int_0^s f_n(t)\,dt\right) = \varphi\left(\int_0^s f(t)\,dt\right)$ *for $0 \leqq s \leqq 2\pi$ and each $\varphi \in X^*$.*

REMARK: Condition (1) implies that $\int_0^{2\pi} ||f_n(t)||\,dt$ is a bounded sequence.

Also (1) implies and is implied by the corresponding condition with f_n replaced by $f - f_n$. Hence for the proof we may as well assume that $f(t) \equiv 0$.

We consider first the sufficiency. It follows from (2) [with $f(t) \equiv 0$] that

$$\int_0^{2\pi} \varphi(t) f_n(t)\,dt \to 0$$

if $\varphi(t)$ is a step-function. Now since X^* satisfies condition (D) the most general linear functional $L\{X\}$ has the form (3.4). If $\varphi(t) \in M\{X^*\}$ we choose a sequence $\{\varphi_i(t)\}$ of step-functions, uniformly bounded and converging to $\varphi(t)$ a.e. Then given $\delta > 0$ we can find a set E of measure $> 2\pi - \delta$ on which $\varphi_i(t)$ converges *uniformly*[15] to $\varphi(t)$. Let ϵ be given and choose δ in accordance with condition (1), where S is the set complementary to E. Finally, choose i so large that $||\varphi_i(t) - \varphi(t)|| < \epsilon$ on the set E. Then

$$\left|\int_0^{2\pi} \varphi(t) f_n(t)\,dt\right| \leqq \left|\int_E [\varphi(t) - \varphi_i(t)] f_n(t)\,dt\right|$$

$$+ \left|\int_S [\varphi(t) - \varphi_i(t)] f_n(t)\,dt\right| + \left|\int_0^{2\pi} \varphi_i(t) f_n(t)\,dt\right|$$

$$\leqq \epsilon \int_0^{2\pi} ||f_n(t)||\,dt + 2M\epsilon + \left|\int_0^{2\pi} \varphi_i(t) f_n(t)\,dt\right|,$$

where M is a common essential bound for $\varphi(t)$ and $\varphi_i(t)$ on the interval. But the integrals $\int_0^{2\pi} ||f_n(t)||\,dt$ are bounded and the last term on the right of the inequality approaches zero with $1/n$. Hence f_n converges weakly to $f \equiv 0$, as was to be proved.

For the proof of the necessity we require the following lemma.

[15] Bochner I, p. 264.

LEMMA. *Given* $\epsilon > 0$, $f(t) \in L\{X\}$, *and a measurable set* E *on* $(0, 2\pi)$ *there exists a function* $\varphi(t) \in M\{X^*\}$ *such that*

a) $||\varphi(t)|| = 1$ *if* $t \in E$;

b) $\varphi(t) = 0$ *if* t *is not in* E;

c) $$\left| \int_E \varphi(t)\, f(t)\, dt \right| \geqq \int_E ||f(t)||\, dt - \epsilon.$$

PROOF: Choose a sequence $\{f_n\}$ of step functions so that $f_n \to f$ in the metric of $L\{X\}$. Then $\int_E ||f(t) - f_n(t)||\, dt \to 0$ and $\int_E ||f_n(t)||\, dt \to \int_E ||f(t)||\, dt$. For a given n, and $t \in E$ choose $\varphi_n(t)$ so that $||\varphi_n(t)|| = 1$ and $\varphi_n(t)\, f_n(t) = ||f_n(t)||$.[16] Define $\varphi_n(t) = 0$ of t is not in E. Since f_n is a step-function we may make our choice so that φ_n is also a step function. Then

$$\left| \int_E \varphi_n(t) f(t)\, dt \right| = \int_E [\varphi_n(t) f(t) - \varphi_n(t) f_n(t)]\, dt + \int_E \varphi_n(t) f_n(t)\, dt \Bigg|$$

$$\geqq \int_E ||f_n(t)||\, dt - \int_E ||f(t) - f_n(t)||\, dt.$$

Since the right member of this inequality approaches $\int_E ||f(t)||\, dt$ we may choose for our $\varphi(t)$ a definite $\varphi_n(t)$ for a sufficiently large value of n.

To return to the proof of Theorem 4.2, it is clear that condition (2) is necessary. Suppose that f_n converges weakly to zero, but that (1) is not true. Then for some $\epsilon_0 > 0$ there exists a sequence of sets $\{E_\nu\}$ with measure $|E_\nu| \to 0$, and a sequence $\{n_\nu\}$ such that

$$\int_{E_\nu} ||f_{n_\nu}(t)||\, dt > \epsilon_0, \qquad \nu = 1, 2, \cdots. \tag{4.1}$$

By a suitable thinning out process we may assume that the E_ν are non-overlapping sets.[17] For convenience we shall write f_ν in place of f_{n_ν}. We shall define a function $\varphi(t) \in M\{X^*\}$ for which $\int_0^{2\pi} \varphi(t)\, f_n(t)\, dt$ does not approach zero, thus contradicting the hypothesis of weak convergence.

To do this choose $\delta_1 > 0$ so that $\int_S ||f_1(t)||\, dt < \epsilon_0/3$ if $|S| \leqq \delta_1$. Then choose $\varphi_1(t)$ according to the specifications a) – c) of the lemma, where $\epsilon = \epsilon_0/6$, $f = f_1$, and $E = E_1$. Thus $\left| \int_{E_1} \varphi_1(t) f_1(t)\, dt \right| \geqq 5\epsilon_0/6$ by (4.1). Now by hypothe-

[16] For the possibility of this choice see Banach, loc. cit. p. 55, Theorem 3, and the reference in footnote (11).

[17] Compare with Lebesgue, Annales de la Faculté des Sciences de Toulouse, tome 1 (1909) esp. pp. 53 and 58.

sis $\int_{E_1} \varphi_1(t) f_n(t)\,dt \to 0$. We choose m_2 so that for $n \geqq m_2$ $\left|\int_{E_1} \varphi_1(t) f_n(t)\,dt\right| < \epsilon_0/3$ and $|E_{m_2} + E_{m_2+1} + \cdots| < \delta_1$. Write $e_1 = E_1$, $m_1 = 1$, $e_2 = E_{m_2}$ and choose δ_2 so that $\int_S ||f_{m_2}(t)||\,dt < \epsilon_0/3$ if $|S| \leqq \delta_2$. Choose $\varphi_2(t)$ by the lemma so that $||\varphi_2(t)|| = 1$ on e_2, $= 0$ elsewhere, and

$$\left|\int_{e_2} \varphi_2(t) f_{m_2}(t)\,dt\right| \geqq \int_{e_2} ||f_{m_2}(t)||\,dt - \epsilon_0/6 \geqq 5\epsilon_0/6.$$

Then $\int_{e_1} \varphi_1(t) f_n(t)\,dt + \int_{e_2} \varphi_2(t) f_n(t)\,dt \to 0$ and we choose m_3 so that for $n \geqq m_3$

$$\left|\int_{e_1} \varphi_1(t) f_n(t)\,dt + \int_{e_2} \varphi_2(t) f_n(t)\,dt\right| < \epsilon_0/3$$

and also $|E_{m_3} + E_{m_3+1} + \cdots| < \delta_2$. Continuing in this way we define $\{m_i\}$, $\{e_i = E_{m_i}\}$, $\{\delta_i\}$ so that $\int_S ||f_{m_i}(t)||\,dt < \epsilon_0/3$ if $|S| \leqq \delta_i$, $\varphi_i(t)$ so that $||\varphi_i(t)|| = 1$ in e_i, $= 0$ elsewhere,

$$\left|\int_{e_i} \varphi_i(t) f_{m_i}(t)\,dt\right| \geqq 5\epsilon_0/6, \quad \left|\int_{e_1} \varphi_1(t) f_n(t)\,dt + \cdots + \int_{e_i} \varphi_i(t) f_n(t)\,dt\right| < \epsilon_0/3,$$

if $n \geqq m_{i+1}$, and $|E_{m_{i+1}} + E_{m_{i+1}+1} + \cdots| < \delta_i$. Then if we define $\varphi(t) = \varphi_i(t)$ in e_i, $\varphi(t) = 0$ elsewhere, $\varphi(t) \in M\{X^*\}$, $||\varphi(t)|| = 1$ in $\sum_i e_i$, $\varphi(t) = 0$ elsewhere. This implies that

$$\left|\int_{e_1+\cdots+e_{i-1}} \varphi(t) f_{m_i}(t)\,dt\right| < \epsilon_0/3,$$

$$\left|\int_{e_{i+1}+e_{i+2}+\cdots} \varphi(t) f_{m_i}(t)\,dt\right| \leqq \int_{e_{i+1}+e_{i+2}+\cdots} ||f_{m_i}(t)||\,dt < \epsilon_0/3,$$

for $|e_{i+1} + e_{i+2} + \cdots| < \delta_i$. But then

$$\left|\int_0^{2\pi} \varphi(t) f_{m_i}(t)\,dt\right| \geqq \left|\int_{e_j} \varphi_i(t) f_{m_i}(t)\,dt\right| - \left|\int_{e_1+\cdots+e_{i-1}} \varphi(t) f_{m_i}(t)\,dt\right| - \left|\int_{e_{i+1}+e_{i+2}+\cdots} \varphi(t) f_{m_i}(t)\,dt\right|$$

$$\geqq \frac{5\epsilon_0}{6} - \frac{\epsilon_0}{3} - \frac{\epsilon_0}{3} = \frac{\epsilon_0}{6} > 0,$$

which is a contradiction. Hence the conditions are necessary.

In order to show that it is essential for the space X^* to satisfy condition (D) in Theorems 4.1 and 4.2 we choose for X the space (l). Then X^* is equivalent

to the same (m),[18] and it is known that (m) fails to satisfy condition (D).[19] For the sequence $\{f_n\} \in L^p\{X\}$ we choose the following functions: $f_n(t) = (f_n^{(1)}(t), f_n^{(2)}(t), \cdots)$ where $f_n^{(\nu)}(t) = \delta_n^\nu \cos nt$, δ_n^ν being the Kronecker symbol. Then $f_n(t)$ is measurable (in fact continuous), and therefore in $L^p\{X\}$, $p \geqq 1$. Moreover $\| f_n(t) \| = |\cos nt| \leqq 1$. Finally

$$\left\| \int_0^s f_n(t)\, dt \right\| = \left| \frac{1}{n} \sin ns \right| \leqq \frac{1}{n}$$

so that the left member of the above inequality tends to zero with $1/n$. Accordingly $\{f_n\}$ satisfies the conditions of both the foregoing theorems (where $f(t) \equiv 0$). Nevertheless the sequence does not converge weakly to zero, as we shall prove by exhibiting a linear functional U such that $U(f_n)$ does not tend to zero.

To define U take a sequence $\{g^{(\nu)}(t)\}$ of numerical functions (measurable) such that $|g^{(\nu)}(t)| \leqq 1$. If $f(t) = (f^{(1)}(t), f^{(2)}(t), \cdots)$ belongs to $L^p\{X\}$ then $f^{(\nu)}(t) \in L^p$. Hence the series $\sum_{\nu=1}^\infty g^{(\nu)}(t) f^{(\nu)}(t)$ converges a.e. and since $|\sum_{\nu=1}^\infty g^{(\nu)}(t) f^{(\nu)}(t)| \leqq \| f(t) \|$ the series defines an integrable function, whose integral we define to be $U(f)$. Then if $p > 1$

$$|U(f)| = \left| \int_0^{2\pi} \left(\sum_1^\infty{}_\nu\, g^{(\nu)}(t) f^{(\nu)}(t) \right) dt \right|$$

$$\leqq \int_0^{2\pi} \| f(t) \|\, dt \leqq (2\pi)^{1/q} \left(\int_0^{2\pi} \| f(t) \|^p\, dt \right)^{1/p},$$

so that U is a linear functional on $L^p\{X\}$, $p \geqq 1$. Now we select $g^{(\nu)}(t)$ in particular to be $g^{(\nu)}(t) = \cos \nu t / |\cos \nu t|$ if $\cos \nu t \neq 0$, $g^{(\nu)}(t) = 0$ otherwise. Therefore $U(f_n) = \int_0^{2\pi} |\cos nt|\, dt = 4$, which gives the desired result.

We turn now to $C\{X\}$.

THEOREM 4.3. *In order that a sequence $\{f_n\}$ of elements of $C\{X\}$ converge weakly to an element $f \in C\{X\}$ it is necessary and sufficient that*

a) $\| f_n(t) \| \leqq A$ *for some constant* A,

b) $\lim_{n \to \infty} \varphi(f_n(t)) = \varphi(f(t))$ *for each* t, $0 \leqq t \leqq 2\pi$, *and each* $\varphi \in X^*$.

As in the discussion of the previous theorem it may be observed that there is no loss of generality in assuming that $f(t) \equiv 0$, so we shall do this. The conditions are clearly necessary. The proof that they are also sufficient depends on a lemma of general nature, which we consider first.

LEMMA. *Let X be a separable space. Let $\Phi(t) \in V^\infty\{X^*\}$. Then there exists a function $\varphi(t)$ defined a.e. with values in X^* such that for each $x \in X$ $\varphi(t)x$ is integrable, and*

$$\int_0^s \varphi(t)x\, dt = \Phi(s)x - \Phi(0)x, \qquad 0 \leqq s \leqq 2\pi. \tag{4.2}$$

[18] Banach, loc, cit. p. 67.

[19] See the paper of Clarkson referred to in footnote 4).

PROOF: Choose a linearly independent, denumerable, everywhere dense set $\{x_n\}$ in X. Since $|\Phi(t+h)x_n - \Phi(t)x_n| \leqq N(\Phi)\,||x_n||$ we know that except on a certain set S_n of measure zero the function $\Phi(t)x_n$ has a derivative $\psi_n(t)$, of which it is the indefinite integral. Let E be the set remaining after discarding $(S_1 + S_2 + \cdots)$. E is of measure 2π. Let X_0 be the (not closed) linear manifold determined by the set $\{x_n\}$. For each $t \epsilon E$ we define a functional $\varphi_0(t)$ on X_0 by the equations $\varphi_0(t)x = \sum_1^n a_r\psi_r(t)$ if $x = \sum_1^n a_r x_r$. Then $\varphi_0(t)x = d/dt\,\Phi(t)x$, and $|\varphi_0(t)x| \leqq N(\Phi)\,||x||$, so that $\varphi_0(t)$ is linear. But then, because X_0 is everywhere dense in X, $\varphi_0(t)$ admits, for each $t \epsilon E$, a unique linear extension[10] $\varphi(t)$ defined throughout X, with the same norm. It is clear that $\varphi(t)x$ is integrable. We must verify that (4.2) is satisfied. It is satisfied if $x \epsilon X_0$, for then $\varphi_0(t)x = \varphi(t)x$. In general, if $x \epsilon X$, and $y_n \to x$, $y_n \epsilon X_0$, then $\Phi(s)x - \Phi(0)x = \lim_{n\to\infty}[\Phi(s)y_n - \Phi(0)y_n] = \lim_{n\to\infty}\int_0^s \varphi(t)y_n\,dt$. But $\varphi(t)y_n \to \varphi(t)x$ a.e. and since $|\varphi(t)y_n| \leqq N(\Phi)\,||y_n||$ it follows by the convergence theorem of Lebesgue that $\int_0^s \varphi(t)y_n\,dt \to \int_0^s \varphi(t)x\,dt$, which is what we wish to prove.

We now return to the proof of Theorem 4.3. By Theorem 3.1 we have to prove that given $\Phi(t) \epsilon V\{X^*\}$ then

$$\lim_{n\to\infty}\int_0^{2\pi} d\Phi(t)f_n(t) = 0.$$

Let us introduce a function $v(t)$, the total variation of $\Phi(t)$ over the interval $(0, t)$. v is a non-negative, bounded, non-decreasing function such that[20]

$$||\Phi(t_1) - \Phi(t_2)|| \leqq |v(t_1) - v(t_2)|$$

v has at most a denumerable set of discontinuities, and its points of continuity are among those of $\Phi(t)$. Let $a = v(2\pi) = V(\Phi)$. We introduce a parameter τ on the interval $(0, a)$ and define a function $\Psi(\tau)$ as follows: $\Psi(\tau) = \Phi(t)$ if $\tau = v(t)$ and v is continuous at t. Otherwise let

$$\Psi(\tau) = \Phi(t-0) + \frac{\Phi(t+0) - \Phi(t-0)}{v(t+0) - v(t-0)}(\tau - v(t-0))$$

in the interval $v(t-0) \leqq \tau \leqq v(t+0)$, thus making Ψ linear there. Then $||\Psi(\tau_1) - \Psi(\tau_2)|| \leqq |\tau_1 - \tau_2|$ so that $\Psi \epsilon V^\infty\{X^*\}$ [for the interval $(0, a)$]. If $f(t) \epsilon C\{X\}$ we define a function $g(\tau)$ on $(0, a)$ by putting $g(\tau) = f(t)$ if $\tau = v(t)$ and v is continuous at t or if $v(t-0) \leqq \tau \leqq v(t+0)$. Then $g(\tau)$ is continuous. From these definitions we infer at once that

$$\int_0^{2\pi} d\Phi(t)f(t) = \int_0^a d\Psi(\tau)g(\tau).$$

[20] The use of this device was suggested by the paper of Dunford and Morse referred to in footnote 4).

Now if X is separable we see from the lemma that there exists a function $\psi(\tau)$ defined a.e. on $(0, a)$ such that $\psi(t)x$ is measurable, and

$$\int_\alpha^\beta \psi(\tau)x\, d\tau = \Psi(\beta)x - \Psi(\alpha)x.$$

Since $\| \psi(\tau) \| \leqq 1$ a.e. it is easily proved that $\psi(\tau)g(\tau)$ is integrable and that

$$\int_0^a d\Psi(\tau)g(\tau) = \int_0^a \psi(\tau)g(\tau)\, d\tau.$$

We have now to prove that

$$\lim_{n\to\infty} \int_0^a \psi(\tau)g_n(\tau)\, d\tau = 0,$$

where $g_n(\tau)$ is defined in terms of $f_n(t)$ as described above. By condition a) we have $| \psi(\tau)g_n(\tau) | \leqq A$. But by condition b) $\psi(\tau)g_n(\tau) = \varphi(t)f_n(t) \to 0$ a.e. and so we are through if X is separable.

If X is not separable we consider the totality of values of $f_n(t)$, $0 \leqq t \leqq 2\pi$, $n = 1, 2, \cdots$, and the closed linear manifold X_0 determined by this set. Since $f_n(t)$ is continuous it is easy to see that X_0 is separable. But any linear functional U on $C\{X\}$ is also a functional on $C(X_0)$, and so, by the above reasoning $U(f_n) \to 0$. This completes the proof.

It will be observed that the conditions stated in the foregoing theorems all have two things in common. In the first place it is assumed that the norms $\| f_n \|$ are bounded, and second, that $U(f_n) \to U(f)$ for all functionals U of a certain class (the step functions in the conjugate space). The question arises as to whether the mere assumption that $U(f_n) \to U(f)$ for all U of some proper subclass of the conjugate space is enough to insure the weak convergence of the sequence f_n to f. That such a situation can arise is shown by the following theorem, where the functionals required by the hypothesis need not even be everywhere dense in the conjugate space.

THEOREM 4.4. *If $f_n(t)$ is a sequence of elements of $C\{X\}$ such that*

$$\lim_{n\to\infty} \int_0^{2\pi} d\Phi(t)f_n(t) = 0 \tag{4.3}$$

for all pure jump functions $\Phi(t) \in V\{X^*\}$ then the sequence $\| f_n(t) \|$ is uniformly bounded and therefore (4.3) *is also true for all $\Phi(t) \in V\{X^*\}$.*

PROOF: Suppose that $\| f_n(t) \|$ is unbounded. Then there will exist a sequence of points t_n such that $\| f_n(t_n) \| \to \infty$. Since the functions are continuous we may assume that the points t_n are all different, and by choosing if necessary a subsequence we may assume that

$$\| f_n(t_n) \| > n. \tag{4.4}$$

Now let $A_n = \max_t \| f_n(t) \|$. We may suppose that $A_n \leqq A_{n+1}$. We construct

a sequence of indices $\{m_\nu\}$ and a sequence $\{\varphi_\nu\}$ of elements of X^* with the following properties:

(a) $$\|\varphi_\nu\| = 1/m_\nu, \quad |\varphi_\nu f_{m_\nu}(t_{m_\nu})| > 1;$$

(b) $$\sum_{i=1}^{\nu-1} |\varphi_i f_{m_\nu}(t_{m_i})| < 1/3, \quad \sum_{i=\nu+1}^{\infty} |\varphi_i f_{m_\nu}(t_{m_i})| < 1/3;$$

(c) $$\sum_{i=1}^{\infty} 1/m_\nu < \infty$$

Suppose for a moment that we have done this. Then we define a pure jump function $\Phi(t)$:

$$\Phi(t) = \sum_{t_{m_\nu} \leqq t} \varphi_\nu.$$

Its total variation is $V(\Phi) = \sum_1^\infty \|\varphi_\nu\|$, which is finite, by (a) and (c). We have, for any $\nu \geqq 2$

$$\left|\int_0^{2\pi} d\Phi(t) f_{m_\nu}(t)\right| = \left|\sum_{i=1}^{\infty} \varphi_i f_{m_\nu}(t_{m_i})\right| \geqq |\varphi_\nu f_{m_\nu}(t_{m_\nu})| - \sum_{i=1}^{\nu-1} |\varphi_i f_{m_\nu}(t_{m_i})| - \sum_{i=\nu+1}^{\infty} |\varphi_i f_{m_\nu}(t_{m_i})| > \tfrac{1}{3}$$

by (b). Since this contradicts (4.3) the proof is completed if we can establish (a), (b), (c).

Begin by taking $m_1 = 1$ and choosing[16] φ_1 so that $\|\varphi_1\| = 1$ and $|\varphi_1 f_1(t_1)| = \|f_1(t_1)\| > 1$ [see (4.4)]. Next choose m_2 so that $m_2 > 2^2 \cdot 3A_{m_1}$ and $|\varphi_1 f_n(t_1)| < \frac{1}{3}$ if $n \geqq m_2$. This can be done since $\varphi f_n(t) \to 0$ for any fixed φ, t by virtue of (4.3). φ_2 is then chosen with $\|\varphi_2\| = 1/m_2$, $|\varphi_2 f_{m_2}(t_{m_2})| = (1/m_2)\|f_{m_2}(t_{m_2})\| > 1$. If $m_1, \cdots, m_{\nu-1}$, $\varphi_1, \cdots, \varphi_{\nu-1}$ have been chosen, choose m_ν so that $m_\nu > 2^\nu \cdot 3A_{m_{\nu-1}}$ and $|\varphi_1 f_n(t_1)| + \cdots + |\varphi_{\nu-1} f_n(t_{m_{\nu-1}})| < \frac{1}{3}$ if $n \geqq m_\nu$; then choose φ_ν with $\|\varphi_\nu\| = 1/m_\nu$ and $|\varphi_\nu f_{m_\nu}(t_{m_\nu})| = (1/m_\nu)\|f_{m_\nu}(t_{m_\nu})\| > 1$. Since $|\varphi_i f_{m_\nu}(t_{m_i})| \leqq A_{m_\nu}/m_i$ and $A_{m_\nu} \leqq A_{m_{i-1}}$ if $i \geqq \nu + 1$ it is clear that all the conditions (a)–(c) are satisfied.

5. Reflexive Spaces

We have already defined the space X^* conjugate to X. We denote by X^{**} the space conjugate to X^*. It is well known that X is equivalent to a closed subspace of X^{**}, namely, the set of all linear functionals G defined on X^* of the form $G(\varphi) = \varphi(x)$ for some $x \in X$.[21]

DEFINITION. *X is said to be reflexive if all the elements of X^{**} have this form, so that X^{**} and X are equivalent.*[22]

---

[21] If G corresponds to x in this manner then by Banach, Theorem 3, p. 55, $\|G\| = \|x\|$.

[22] We prefer the term *reflective* to the adjective *regular* used by Hahn, Journal fur Mathematik, vol. 157 (1927), p. 220.

We shall later need the following result.

THEOREM 5.1. *If X is not reflexive X^* is likewise not reflexive.*

PROOF: Denote by X_1 the subspace of X^{**} which is equivalent to X under the correspondence defined above. Since X_1 is closed, and a proper subspace of X^{**} we can find a functional $\mathcal{F}_0$ defined over X^{**} (i.e. an element of X^{***}) such that $\mathcal{F}_0 \neq 0$ and yet $\mathcal{F}_0(G) = 0$ if $G \in X_1$.[23] Now if X^* were reflexive we could write $\mathcal{F}_0(G) = G(\varphi_0)$ for some $\varphi_0 \neq 0 \in X^*$ and all $G \in X^{**}$. But then $G(\varphi_0) = 0$ if $G \in X_1$—that is $\varphi_0(x) = 0$ if $x \in X$. This contradicts $\varphi_0 \neq 0$ and completes the proof.

Another concept which is closely related to that of a reflexive space is that of weak completeness.

DEFINITION. *X is said to be (sequentially) weakly complete*[24] *if whenever $\{x_n\}$ is a sequence such that $\lim_{n\to\infty} \varphi(x_n)$ exists for each $\varphi \in X^*$ then there exists an $x \in X$ for which $\lim_{n\to\infty} \varphi(x_n) = \varphi(x)$.*

It is well known that every reflexive space is weakly complete. Now if X and X^* satisfy condition (D) and if X is reflexive it follows from the results of §2, 3 that $L^p\{X\}$ is reflexive if $p > 1$, and hence weakly complete. It will be shown in §7 that if $L^p\{X\}$ is reflexive then X must also be reflexive and X, X^* must satisfy condition (D). Under these hypotheses on X, X^* we shall now show that $L\{X\}$ is weakly complete in the above sense.

THEOREM 5.2. *If X is reflexive and if X, X^* both satisfy condition* (D) *then $L\{X\}$ is weakly complete.*

PROOF: Let $\{f_n\}$ be a sequence in $L\{X\}$ and suppose that $\lim_{n\to\infty} \int_0^{2\pi} \varphi(t) f_n(t)\, dt$ exists for each $\varphi(t) \in M\{X^*\}$. Let us write

$$G_n(\varphi) = \int_0^{2\pi} \varphi(t) f_n(t)\, dt, \qquad G(\varphi) = \lim_{n\to\infty} G_n(\varphi).$$

Then G_n, G are elements of $[M\{X^*\}]^*$. We have to prove that G has the form

$$G(\varphi) = \int_0^{2\pi} \varphi(t) f(t)\, dt$$

for some $f(t) \in L\{X\}$.

For fixed φ, s, $\varphi \in X^*$, $0 \leqq s \leqq 2\pi$, define $H_n(s)\varphi = \int_0^s \varphi f_n(t)\, dt$, $H(s)\varphi = \lim_{n\to\infty} H_n(s)\varphi$. The function $H(s)$ thus defined on $(0, 2\pi)$ to X^{**} is absolutely continuous—that is $H(s) \in AC_0\{X^{**}\}$. To prove this take a set of non-over-

[23] Banach, loc. cit. p. 57.

[24] Banach, loc, cit. p. 140. We emphasize the adjective *sequentially* to distinguish this concept from a more far-reaching concept which has also been called weak completeness. See H. H. Goldstine, Duke Mathematical Journal, vol. 4 (1938), p. 123–131. We use the term in Banach's sense only.

lapping intervals $(t_\nu, t_\nu + h_\nu)$. Let $\epsilon > 0$ be given and choose $\delta > 0$ so that $\int_S \| f_n(t)\, dt < \frac{1}{2}\epsilon$, $n = 1, 2, \cdots$ if $|S| < \delta$. The possibility of doing this follows by a simple argument from Theorem 4.2. Now if $\sum_\nu h_\nu < \delta$ and S denotes the above set of intervals we have, for $\| \varphi_\nu \| = 1$

$$\sum_\nu | H_n(t_\nu + h_\nu)\varphi_\nu - H_n(t_\nu)\varphi_\nu | = \sum_\nu \left| \int_{t_\nu}^{t_\nu + h_\nu} \varphi_\nu f_n(t)\, dt \right|$$

$$\leqq \sum_\nu \int_{t_\nu}^{t_\nu + h_\nu} \| f_n(t) \| \, dt = \int_S \| f_n(t) \| \, dt < \epsilon/2.$$

Hence, letting $n \to \infty$

$$\sum_\nu | H(t_\nu + h_\nu)\varphi_\nu - H(t_\nu)\varphi_\nu | \leqq \epsilon/2.$$

But we can choose φ_ν so that

$$| H(t_\nu + h_\nu)\varphi_\nu - H(t_\nu)\varphi_\nu | > \| H(t_\nu + h_\nu) - H(t_\nu) \| - \epsilon/2m,$$

where m is the number of intervals. Thus

$$\sum_\nu \| H(t_\nu + h_\nu) - H(t_\nu) \| \leqq \epsilon,$$

which proves that $H(t) \in AC_0\{X^{**}\}$.

Now since X is reflexive there exists a function $F(t) \in AC_0\{X\}$ such that $H(t)\varphi = \varphi(F(t))$ for each $\varphi \in X^*$ and each t. But since X satisfies condition (D), we know that $F'(t)$ exists a.e. and is in $L\{X\}$, with

$$F(s) = \int_0^s F'(t)\, dt.$$

Thus

$$\lim_{n \to \infty} \varphi \left(\int_0^s f_n(t)\, dt \right) = \varphi \left(\int_0^s F'(t)\, dt \right),$$

whence, by Theorem 4.2, f_n converges weakly to F'.

In the opposite direction we have the easier theorem:

THEOREM 5.3. *If $L\{X\}$ is weakly complete so is X.*

PROOF: Suppose that $\{x_n\}$ is a sequence in X such that $\lim_{n \to \infty} \varphi(x_n)$ exists for each $\varphi \in X^*$. Define $f_n(t) = x_n$. Then $f_n \in L\{X\}$. For any linear functional U on $L\{X\}$

$$U(f_n) = \int_0^{2\pi} d\Phi(t) f_n(t) = \Phi(2\pi) x_n,$$

where $\Phi(t) \in V_0^\infty\{X^*\}$. Hence $\lim_{n \to \infty} U(f_n)$ exists. But since $L(X)$ is weakly complete there exists a function $f(t) \in L\{X\}$ such that

$$\Phi(2\pi) x_n \to \int_0^{2\pi} d\Phi(t) f(t),$$

for all $\Phi(t) \in V_0^\infty\{X^*\}$. Hence in particular if $\varphi \in X^*$ and $\Phi(t) = t\varphi$.

$$2\pi\varphi(x_n) \to \varphi\left(\int_0^{2\pi} f(t)\,dt\right),$$

whence, if we put $x = (\frac{1}{2}\pi)\int_0^{2\pi} f(t)\,dt$, the proof is complete.

6. Fourier Series

It was noted in Bochner I that with the abstract Lebesgue integral it is possible to carry over much of the theory of Fourier series for abstract functions. We shall consider some aspects of this theory as it applies to our investigations of the spaces $L^p\{X\}$.

The following notation is used throughout. If $f(t) \in L\{X\}$ its Fourier coefficients are defined by

$$f_n = \frac{1}{2\pi}\int_0^{2\pi} e^{-injj} f(t)\,dt \qquad n = 0, \pm 1, \cdots. \tag{6.1}$$

So long as we work in the spaces $L^p\{X\}$ we may suppose that f is defined for almost all real values of t and has the period 2π. The partial sums of the Fourier series for f are then

$$s_n(t) = \sum_{\nu=-n}^{n} e^{i\nu t} f_\nu = \frac{1}{\pi}\int_0^{2\pi} D_n(u) f(t+u)\,du, \tag{6.2}$$

where $D_n(u)$ is the Dirichlet kernel.[25] The Fejér sums (or first Cesàro means) corresponding to f are

$$\sigma_n(t) = \sum_{\nu=-n}^{n}\left(1 - \frac{|\nu|}{n+1}\right)e^{i\nu t} f_\nu = \frac{1}{\pi}\int_0^{2\pi} K_n(u) f(t+u)\,du, \tag{6.3}$$

where $K_n(u)$ is the Fejér kernel.[26] If $\sigma_n(t) \to x$ the Fourier series is said to be summable $(C, 1)$ to the value x at the point t.

We require the following two standard theorems.

THEOREM 6.1. *If $f(t)$, defined for all real t, has the period 2π and belongs to $C\{X\}$ its Fourier series is uniformly summable $(C, 1)$ to $f(t)$ on the interval $(0, 2\pi)$.*

THEOREM 6.2. *If $f(t) \in L^p\{X\}$, $p \geqq 1$ then*

$$\lim_{n\to\infty}\int_0^{2\pi} \|f(t) - \sigma_n(t)\|^p\,dt = 0.$$

These theorems are proved in the usual way.

If $\Phi(t) \in V^p\{X\}$, $1 \leqq p \leqq \infty$ the integrals

$$\Phi_n = \frac{1}{2\pi}\int_0^{2\pi} e^{-int}\,d\Phi(t) \tag{6.4}$$

exist. They are of type (2.3).

---

[25] Zygmund, *Trigonometrical Series*, Warsaw (1935), p. 21.

[26] Zygmund, loc. cit. p. 45.

We are interested in generalizations of Parseval's theorem. The generalization takes the following form.

THEOREM 6.3. *If $f(t)$ has period 2π and belongs to $L^p\{X\}$ and if $\Phi(t) \in V^q\{X^*\}$, $1 < p$, $1/p + 1/q = 1$, then*

$$\frac{1}{2\pi}\int_0^{2\pi} d\Phi(t)f(t) = \sum_{-\infty}^{\infty} \Phi_n(f_{-n}) \qquad (C, 1) \tag{6.5}$$

the series on the right being summable $(C, 1)$ to the expression on the left. If $p = 1$ the same result holds if we put $q = \infty$. The formula (6.5) is valid if $f(t) \in C\{X\}$ and $\Phi(t) \in V\{X^*\}$.

PROOF: We must show that

$$\lim_{n\to\infty} \sum_{\nu=-n}^{n} \left(1 - \frac{|\nu|}{n+1}\right)\Phi_\nu(f_{-\nu}) = \frac{1}{2\pi}\int_0^{2\pi} d\Phi(t)f(t).$$

Consider first the case where $f(t) \in L^p\{X\}$, $\Phi(t) \in V^q\{X^*\}$. Now

$$\frac{1}{2\pi}\int_0^{2\pi} d\Phi(t)\sigma_n(t) = \sum_{\nu=-n}^{n}\left(1 - \frac{|\nu|}{n+1}\right)\frac{1}{2\pi}\int_0^{2\pi} d\Phi(t)e^{i\nu t}f_\nu$$

$$= \sum_{\nu=-n}^{n}\left(1 - \frac{|\nu|}{n+1}\right)\Phi_{-\nu}(f_\nu)$$

by (6.4) and the properties of the Stieltjes integral (2.3). From this we see that it is sufficient to prove that

$$\lim_{n\to\infty} \frac{1}{2\pi}\int_0^{2\pi} d\Phi(t)\sigma_n(t) = \frac{1}{2\pi}\int_0^{2\pi} d\Phi(t)f(t). \tag{6.6}$$

But this is true as a result of Theorem 6.2, for (6.6) merely expresses the continuity property, that $U(\sigma_n) \to U(f)$, where U is the functional on $L^p\{X\}$ defined by $\Phi(t)$. The conclusion for $f(t) \in C\{X\}$, $\Phi(t) \in V\{X^*\}$ follows in a similar way from Theorem 6.1.

In the theory of ordinary Fourier series it is known that the theorem corresponding to Theorem 6.3 for $1 < p < \infty$ is true even with the stronger assertion that the series on the right in (6.5) is convergent.[27] Now it may be seen from considerations similar to those in the proof just finished that (6.5) will be true, in the sense that the series is convergent, instead of merely summable $(C, 1)$ if and only if

$$\lim_{n\to\infty} \frac{1}{2\pi}\int_0^{2\pi} d\Phi(t)s_n(t) = \frac{1}{2\pi}\int_0^{2\pi} d\Phi(t)f(t). \tag{6.7}$$

But this relation, for a fixed $f(t)$ and *all* $\Phi(t)$, means that the partial sums s_n *converge weakly* to the element f in the space under consideration. This in turn implies that the norms $\| s_n \|$ are bounded. Concerning this matter we have the following theorem.

[27] Zygmund, loc. cit. p. 154.

THEOREM 6.4. *If for each $f(t) \in L^p\{X\}$, $p > 1$, the partial sums $s_n(t)$ of the corresponding Fourier series have the property*

$$\overline{\lim_{n\to\infty}} \left(\int_0^{2\pi} || s_n(t) ||^p \, dt \right) < \infty,$$

then

$$\lim_{n\to\infty} \int_0^{2\pi} || f(t) - s_n(t) ||^p \, dt = 0.$$

PROOF: We regard s_n as being derived from f by an operation T_n defined in $L^p\{X\}$ by the equation (6.2). T_n is linear, for it is obviously additive, and if we apply a standard inequality[28] to (6.2) we get

$$|| T_n f || = \left(\int_0^{2\pi} || s_n(t) ||^p \, dt \right)^{1/p} \leqq \frac{1}{\pi} \int_0^{2\pi} | D_n(u) | \, du \left(\int_0^{2\pi} || f(t) ||^p \, dt \right)^{1/p}.$$

Now by hypothesis $\overline{\lim_{n\to\infty}} || T_n f || < \infty$. Hence by a theorem of Banach and Steinhaus[29] there exists a constant A such that $|| T_n f || \leqq A || f ||$ if $f \in L^p\{X\}$.

Let $\epsilon > 0$ be given and choose N so large that $|| f - \sigma_N || < \epsilon$. This we can do by Theorem 6.2. Write $f - \sigma_N = g$. Now if n is sufficiently large $T_n \sigma_N = \sigma_N$, $T_n f - f = T_n g + \sigma_N - f$, and so $|| T_n f - f || \leqq A\epsilon + \epsilon$, which proves that $T_n f = s_n \to f$ (convergence in the metric of $L^p\{X\}$).

We shall now prove by counter-examples that there exist spaces X (namely M, (m), L, (l)) such that the hypotheses of Theorem 6.4 are not satisfied for any $p > 1$. *For such spaces it follows that for at least one $f \in L^p\{X\}$ the partial sums s_n do not converge weakly to f, and therefore* (6.5) *is not in general true with the requirement that the series converge.*

For X take the space M. Let $x_0(s)$ be a continuous numerical function with period 2π. Define a function $f(t)$ on $(0, 2\pi)$ to M by the following rule: to the number t corresponds the function $x(s) = x_0(s + t)$ in M. Since $x_0(s)$ is periodic it is uniformly continuous on the infinite interval $-\infty < s < \infty$. But

$$|| f(t_1) - f(t_2) || = \sup_{0 \leqq s \leqq 2\pi} | x_0(s + t_1) - x_0(s + t_2) |,$$

and so we infer that $f(t)$ is continuous. It is also periodic. The Fourier coefficients of f are elements of M. They are

$$fn = \frac{1}{2\pi} \int_0^{2\pi} e^{-int} x_0(s + t) \, dt = \frac{1}{2\pi} e^{ins} \int_0^{2\pi} e^{-inu} x_0(u) \, du.$$

[28] The inequality in question is

$$\left\{ \int_a^b \left| \int_c^d h(s, t) \, dt \right|^r ds \right\}^{1/r} \leqq \int_c^d \left\{ \int_a^b | h(s, t) |^r \, ds \right\}^{1/r} dt, \qquad r \geqq 1.$$

See Zygmund, loc. cit. p. 67.

[29] Banach, loc. cit. p. 80, Theorem 5.

Thus

$$s_n(t) = \sum_{\nu=-n}^{n} d^{i\nu(s+t)} \frac{1}{2\pi} \int_0^{2\pi} e^{-i\nu u} x_0(u)du = x_0^{(n)}(s+t)$$

where $x_0^{(n)}(s)$ denotes the n^{th} partial sum of the Fourier series of the function $x_0(s)$. Therefore

$$|| s_n(t) = \sup_{0\leq s\leq 2\pi} |x_0^{(n)}(s+t)| = \sup_{-\infty<s<\infty} |x_0^{(n)}(s)|.$$

In particular, then, $|| s_n(t) ||$ is independent of t, and in order to have

$$\overline{\lim_{n\to\infty}} \int_0^{2\pi} || s_n(t) ||^p \, dt = \infty \qquad p > 1, \tag{6.8}$$

it is sufficient to choose $x_0(s)$ so that the partial sums of its Fourier series are unbounded. That such continuous functions exist is well known.

The foregoing example can easily be modified so as to apply if X is the sequence space (m). To do this we first choose a denumerable set $\{\tau_n\}$ everywhere dense on $(0, 2\pi)$. Then with the above function $x_0(s)$ define the function $f(t)$ on $(0, 2\pi)$ to (m) with components $f_i(t)$ by the equations $f_i(t) = x_0(\tau_i + t)$. Since a continuous function is determined by its values on an everywhere dense set the previous argument is not essentially affected.

In order to reach the situation expressed by (6.8) when X is either of the spaces L, (l) we proceed indirectly, making use of the counter-examples just constructed for the spaces M, (m), which are equivalent, respectively, to the conjugate spaces of L, (l).[30] Suppose then that with this choice of X the hypothesis of Theorem 6.4 is satisfied. Then

$$\frac{1}{2\pi} \int_0^{2\pi} d\Phi(t)f(t) = \lim_{n\to\infty} \frac{1}{2\pi} \int_0^{2\pi} d\Phi(t)s_n(t) = \lim_{n\to\infty} \sum_{\nu=-n}^{n} \Phi_\nu(f_{-\nu}),$$

for each $f(t) \in L^p\{X\}$ and each $\Phi(t) \in V^q\{X^*\}$. But

$$\sum_{\nu=-n}^{n} \Phi_\nu(f_{-\nu}) = \frac{1}{2\pi} \int_0^{2\pi} \varphi_n(t)f(t)\, dt, \tag{6.9}$$

where $\varphi_n(t) = \sum_{-n}^{n} e^{i\nu t} \Phi_\nu$ are elements of $L^q\{X^*\}$ and the integrals $\int_0^{2\pi} \varphi_n(t)f(t)\, dt$ are therefore linear functionals U_n on $L^p\{X\}$ of norm

$$|| U_n || = \left(\int_0^{2\pi} || \varphi_n(t) ||^q \, dt \right)^{1/q}. \tag{6.10}$$

Since we have

$$U_n(f) \to U(f) = \int_0^{2\pi} d\Phi(t)f(t),$$

[30] Banach, loc. cit. p. 65 and p. 67.

the linear functionals U_n converge weakly to the functional U, and therefore the norms $\| U_n \|$ must be bounded.[31] Now let $\varphi(t)$ be a function in $L^q\{X^*\}$ (X^* equivalent to M or (m)) such that the partial sums of its Fourier series are unbounded. Such functions exist, as we saw above. If $\Phi(t)$ is the indefinite integral of $\varphi(t)$, $\Phi(t) \in V^q\{X^*\}$, and the functions $\varphi_n(t) = \sum_{-n}^{n} e^{it\nu}\Phi_\nu$ are precisely the partial sums of the Fourier series for $\varphi(t)$. But then the norms $\| U_n \|$ in (6.10) must be unbounded, which is a contradiction.

We observe[4] that the space (l) satisfies condition (D), which is therefore not enough to guarantee the validity of the hypothesis of Theorem 6.4.

We shall conclude this section with a few theorems about Fourier coefficients. It is known[32] that Bessel's inequality is not in general true—that is, there exist spaces X (in particular the space L) and functions $f(t) \in L\{X\}$ (even in $C\{X\}$) such that $\sum_{-\infty}^{\infty} \| f_n \|^2 = \infty$, where $\{f_n\}$ are the Fourier coefficients of $f(t)$. It is possible, however, to obtain theorems corresponding to the Riesz-Fischer theorem and the theorems of Hausdorff and Young.

THEOREM 6.5. *If $1 < p \leqq 2$ and x_n is a sequence of elements in X ($n = 0, \pm 1, \pm 2, \cdots$) such that for a certain constant $A > 0$ and all elements $\varphi \in X^*$ the inequality*

$$\sum_{-\infty}^{\infty}{}_{\nu} | \varphi(x_\nu) |^p \leqq A^p \| \varphi \|^p \tag{6.11}$$

holds, then there exists a function $F(t) \in C\{X\}$ such that for each $\varphi \in X^*$ $\varphi(F(t))$ is the indefinite integral of a function of class L^q with the Fourier coefficients $\{\varphi(x_n)\}$. The inequality

$$\left(\frac{1}{2\pi}\int_0^{2\pi} \left|\frac{d}{dt}\varphi(F(t))\right|^q dt\right)^{1/q} \leqq \left(\sum_{-\infty}^{\infty}{}_{\nu} | \varphi(x_\nu) |^p\right)^{1/p} \tag{6.12}$$

holds for each φ.

PROOF: The Hausdorff-Young theorem[33] assures us that there exists, corresponding to each φ, a function $f_\varphi(t) \in L^q$ with the Fourier coefficients $\{\varphi(x_n)\}$ and satisfying the inequality

$$\left(\frac{1}{2\pi}\int_0^{2\pi} | f_\varphi(t) |^q dt\right)^{1/q} \leqq \left(\sum_{-\infty}^{\infty}{}_{\nu} | \varphi(x_\nu) |^p\right)^{1/p}.$$

We have then only to show that there exists a function $F(t)$ as described above for which $d/dt\, \varphi(F(t)) = f_\varphi(t)$. For a fixed φ define

$$G_\varphi(t) = \int_0^t [f_\varphi(s) - \varphi(x_0)]\, ds.$$

[31] Banach, loc. cit. p. 122–123, Theorem 2.

[32] Bochner, *Abstrakte Funktionen und die Besselsche Ungleichung*, Göttingen Nachrichten (1933) pp. 178–180.

[33] Zygmund, loc. cit. p. 190, (b).

Then $G_\varphi(t)$ is absolutely continuous and periodic, so that its Fourier series converges uniformly to $G_\varphi(t)$. A calculation shows that

$$\frac{1}{2\pi}\int_0^{2\pi} G_\varphi(t)e^{-int}\,dt = \frac{1}{in}\varphi(x_n), \qquad n \neq 0,$$

and therefore

$$G_\varphi(t) = \sum_{-\infty}^{\infty}{}' \frac{e^{i\nu t}-1}{i\nu}\varphi(x_\nu) \tag{6.13}$$

where the prime denotes omission of the term corresponding to $\nu = 0$. The series in (6.13) converges uniformly as a function of t, φ, for $0 \leqq t \leqq 2\pi$, $\|\varphi\| \leqq 1$, as may be seen by an application of Hölder's inequality and use of (6.11). This implies that the series

$$g(t) = \sum_{-\infty}^{\infty}{}' \frac{e^{i\nu t}-1}{i\nu}x_\nu \tag{6.14}$$

converges uniformly and defines a function $g(t) \in C\{X\}$. For $G_\varphi(t)$ defines a function on $(0, 2\pi)$ to X^{**} which is a uniform limit according to the metric of X^{**} of the sequence of functions in $C\{X_1\}$ defined by the partial sum

$$\sum_{-n}^{n}{}' \frac{e^{i\nu t}-1}{i\nu}\varphi(x_\nu),$$

where X_1 is the *closed* linear subspace of X^{**} equivalent to X, as described in §5. The corresponding functions in $C\{X\}$, which are

$$\sum_{-n}^{n}{}' \frac{e^{i\nu t}-1}{i\nu}x_\nu,$$

therefore converge uniformly to a limit function in $C\{X\}$, as asserted. We see that $\varphi(g(t)) = G_\varphi(t)$. If we put $F(t) = tx_0 + g(t)$, then $\varphi(F(t)) = t\varphi(x_0) + G_\varphi(t)$, $d/dt\,\varphi(F(t)) = f_\varphi(t)$, which completes the proof.

THEOREM 6.6. *If $1 < p \leqq 2$ and $F(t) \in C\{X\}$ is such that for each $\varphi \in X^*$ $\varphi(F(t))$ is the indefinite integral of a function of class L^p, and if for a certain constant $A > 0$ the inequality*

$$\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{d}{dt}\varphi(F(t))\right|^p dt \leqq A^p\,\|\varphi\|^p \tag{6.15}$$

holds, then there exists a sequence $\{x_n\}$ $(n = 0, \pm 1, \cdots,)$ of elements in X such that $\{\varphi(x_n)\}$ are the Fourier coefficients of $d/dt\,\varphi(F(t))$, and

$$\left(\sum_{-\infty}^{\infty}{}_\nu\,|\varphi(x_\nu)|^q\right)^{1/q} \leqq \left(\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{d}{dt}\varphi(F(t))\right|^p dt\right)^{1/p}. \tag{6.16}$$

PROOF: The Hausdorff-Young theorem[34] states that the Fourier coefficients of $d/dt\,\varphi(F(t))$ satisfy an inequality of the form (6.16). Hence all that remains is to show that these coefficients can be put in the required form. Now

[34] Zygmund, loc. cit. p. 190, (a).

the linear functionals U_n converge weakly to the functional U, and therefore the norms $\| U_n \|$ must be bounded.[31] Now let $\varphi(t)$ be a function in $L^q\{X^*\}$ (X^* equivalent to M or (m)) such that the partial sums of its Fourier series are unbounded. Such functions exist, as we saw above. If $\Phi(t)$ is the indefinite integral of $\varphi(t)$, $\Phi(t) \in V^q\{X^*\}$, and the functions $\varphi_n(t) = \sum_{-n}^{n} e^{i\nu t}\Phi_\nu$ are precisely the partial sums of the Fourier series for $\varphi(t)$. But then the norms $\| U_n \|$ in (6.10) must be unbounded, which is a contradiction.

We observe[4] that the space (l) satisfies condition (D), which is therefore not enough to guarantee the validity of the hypothesis of Theorem 6.4.

We shall conclude this section with a few theorems about Fourier coefficients. It is known[32] that Bessel's inequality is not in general true—that is, there exist spaces X (in particular the space L) and functions $f(t) \in L\{X\}$ (even in $C\{X\}$) such that $\sum_{-\infty}^{\infty} \| f_n \|^2 = \infty$, where $\{f_n\}$ are the Fourier coefficients of $f(t)$. It is possible, however, to obtain theorems corresponding to the Riesz-Fischer theorem and the theorems of Hausdorff and Young.

THEOREM 6.5. *If $1 < p \leqq 2$ and x_n is a sequence of elements in X ($n = 0, \pm 1, \pm 2, \cdots$) such that for a certain constant $A > 0$ and all elements $\varphi \in X^*$ the inequality*

$$\sum_{-\infty}^{\infty}{}_{\nu} | \varphi(x_\nu) |^p \leqq A^p \| \varphi \|^p \tag{6.11}$$

holds, then there exists a function $F(t) \in C\{X\}$ such that for each $\varphi \in X^*$ $\varphi(F(t))$ is the indefinite integral of a function of class L^q with the Fourier coefficients $\{\varphi(x_n)\}$. The inequality

$$\left(\frac{1}{2\pi}\int_0^{2\pi} \left|\frac{d}{dt}\varphi(F(t))\right|^q dt\right)^{1/q} \leqq \left(\sum_{-\infty}^{\infty}{}_{\nu} | \varphi(x_\nu) |^p\right)^{1/p} \tag{6.12}$$

holds for each φ.

PROOF: The Hausdorff-Young theorem[33] assures us that there exists, corresponding to each φ, a function $f_\varphi(t) \in L^q$ with the Fourier coefficients $\{\varphi(x_n)\}$ and satisfying the inequality

$$\left(\frac{1}{2\pi}\int_0^{2\pi} | f_\varphi(t) |^q dt\right)^{1/q} \leqq \left(\sum_{-\infty}^{\infty}{}_{\nu} | \varphi(x_\nu) |^p\right)^{1/p}.$$

We have then only to show that there exists a function $F(t)$ as described above for which $d/dt\, \varphi(F(t)) = f_\varphi(t)$. For a fixed φ define

$$G_\varphi(t) = \int_0^t [f_\varphi(s) - \varphi(x_0)]\, ds.$$

---

[31] Banach, loc. cit. p. 122–123, Theorem 2.

[32] Bochner, *Abstrakte Funktionen und die Besselsche Ungleichung*, Göttingen Nachrichten (1933) pp. 178–180.

[33] Zygmund, loc. cit. p. 190, (b).

Then $G_\varphi(t)$ is absolutely continuous and periodic, so that its Fourier series converges uniformly to $G_\varphi(t)$. A calculation shows that

$$\frac{1}{2\pi}\int_0^{2\pi} G_\varphi(t)e^{-int}\,dt = \frac{1}{in}\varphi(x_n), \qquad n \neq 0,$$

and therefore

$$G_\varphi(t) = \sum_{-\infty}^{\infty}{}' \frac{e^{i\nu t}-1}{i\nu}\varphi(x_\nu) \tag{6.13}$$

where the prime denotes omission of the term corresponding to $\nu = 0$. The series in (6.13) converges uniformly as a function of t, φ, for $0 \leqq t \leqq 2\pi$, $\|\varphi\| \leqq 1$, as may be seen by an application of Hölder's inequality and use of (6.11). This implies that the series

$$g(t) = \sum_{-\infty}^{\infty}{}' \frac{e^{i\nu t}-1}{i\nu}x_\nu \tag{6.14}$$

converges uniformly and defines a function $g(t) \in C\{X\}$. For $G_\varphi(t)$ defines a function on $(0, 2\pi)$ to X^{**} which is a uniform limit according to the metric of X^{**} of the sequence of functions in $C\{X_1\}$ defined by the partial sum

$$\sum_{-n}^{n}{}' \frac{e^{i\nu t}-1}{i\nu}\varphi(x_\nu),$$

where X_1 is the *closed* linear subspace of X^{**} equivalent to X, as described in §5. The corresponding functions in $C\{X\}$, which are

$$\sum_{-n}^{n}{}' \frac{e^{i\nu t}-1}{i\nu}x_\nu,$$

therefore converge uniformly to a limit function in $C\{X\}$, as asserted. We see that $\varphi(g(t)) = G_\varphi(t)$. If we put $F(t) = tx_0 + g(t)$, then $\varphi(F(t)) = t\varphi(x_0) + G_\varphi(t)$, $d/dt\,\varphi(F(t)) = f_\varphi(t)$, which completes the proof.

THEOREM 6.6. *If $1 < p \leqq 2$ and $F(t) \in C\{X\}$ is such that for each $\varphi \in X^*$ $\varphi(F(t))$ is the indefinite integral of a function of class L^p, and if for a certain constant $A > 0$ the inequality*

$$\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{d}{dt}\varphi(F(t))\right|^p dt \leqq A^p\,\|\varphi\|^p \tag{6.15}$$

holds, then there exists a sequence $\{x_n\}$ $(n = 0, \pm 1, \cdots,)$ of elements in X such that $\{\varphi(x_n)\}$ are the Fourier coefficients of $d/dt\,\varphi(F(t))$, and

$$\left(\sum_{\infty-}^{\infty}{}_\nu\,|\,\varphi(x_\nu)\,|^q\right)^{1/q} \leqq \left(\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{d}{dt}\varphi(F(t))\right|^p dt\right)^{1/p}. \tag{6.16}$$

PROOF: The Hausdorff-Young theorem[34] states that the Fourier coefficients of $d/dt\,\varphi(F(t))$ satisfy an inequality of the form (6.16). Hence all that remains is to show that these coefficients can be put in the required form. Now

[34] Zygmund, loc. cit. p. 190, (a).

$$\int_0^{2\pi} \frac{d}{dt}\varphi(F(t))\,dt = \varphi(F(2\pi) - F(0)),$$

$$\int_0^{2\pi} e^{-int}\frac{d}{dt}\varphi(F(t))\,dt = [e^{-int}\varphi(F(t))]_0^{2\pi} + in\int_0^{2\pi} e^{-int}\varphi(F(t))\,dt$$

$$= \varphi(F(2\pi) - F(0)) + in\varphi\left(\int_0^{2\pi} e^{-int}F(t)\,dt\right), \qquad (n \neq 0).$$

Hence if we define $\{x_n\}$ by the equations

$$x_0 = \frac{1}{2\pi}(F(2\pi) - F(0)),$$

$$x_n = x_0 + \frac{in}{2\pi}\int_0^{2\pi} e^{-int}F(t)\,dt, \tag{6.17}$$

we see that $\{\varphi(x_n)\}$ are the Fourier coefficients of $d/dt\,\varphi(F(t))$. It may be proved as before that

$$F(t) - F(0) = tx_0 - \sum_{-\infty}^{\infty}{}' \frac{e^{i\nu t} - 1}{i\nu}x_\nu.$$

REMARK: If $p = 2$, $q = 2$ also and the two theorems are converses of each other. In this case the inequality (6.16) becomes an equality.

It is to be emphasized that the derivative $d/dt\,\varphi(F(t))$ need not be expressible in the form $\varphi f(t)$, where $f(t)$ is a function on $(0, 2\pi)$ to X. This is illustrated by the following example. Let X be the space l^2 of sequences $x = \{\xi_n\}$ such that $\sum_{-\infty}^{\infty}{}_\nu |\xi_\nu|^2 = \|x\|^2 < \infty$. Then the functionals $\varphi(x)$ take the form

$$\varphi(x) = \sum_{-\infty}^{\infty}{}_\nu \varphi_\nu\xi_\nu,$$

where $\{\varphi_n\} \in l^2$ also. Corresponding to any such φ there is a function $f_\varphi(t)$ of class L^2 with the Fourier coefficients $\{\varphi_n\}$, and if we put

$$G_\varphi(t) = \int_0^t [f_\varphi(t) - \varphi_0]\,dt,$$

then $G_\varphi(t)$ is absolutely continuous and

$$G_\varphi(t) = \sum_{-\infty}^{\infty}{}' \frac{e^{i\nu t} - 1}{i\nu}\varphi_\nu.$$

Let $F(t)$ be a function on $(0, 2\pi)$ to l^2 defined by the sequence $\{F_n(t)\}$, where $F_n(t) = (e^{int} - 1)/in$, $n \neq 0$, $F_0(t) = t$. This function is well-defined, and $\varphi(F(t)) = t\varphi_0 + G_\varphi(t)$, $d/dt\,\varphi(F(t)) = f_\varphi(t)$. It may be verified by the reader, using Theorem 6.5, that $F(t) \in C(l^2)$. Nevertheless it is impossible to write

$f_\varphi(t)$ in the form $\varphi(f(t))$. For suppose $f(t)$ were given as a sequence $\{f_n(t)\}$ in l^2, so that

$$f_\varphi(t) = \varphi(f(t)) = \sum_{-\infty}^{\infty}{}_\nu \varphi_\nu f_\nu(t).$$

Since $f_\varphi(t)$ has the Fourier series $\sum_{-\infty}^{\infty}{}_\nu \varphi_\nu e^{i\nu t}$ we see that $f_n(t) = e^{int}$. But then $\sum_{-\infty}^{\infty}{}_\nu |f_\nu(t)|^2 = \infty$, so that such a representation is impossible.

7. A Theorem on Reflexive Spaces and the Condition (D)

In §5 it was noted that if X and X^* both satisfy condition (D) and if X is reflexive then $L^p\{X\}$ is reflexive if $p > 1$. We shall now consider a converse proposition.

THEOREM 7.1. *For a fixed $p > 1$ and a fixed space X let $L^p\{X\}$ be reflexive. Then X is reflexive and X, X^* both satisfy condition* (D). *Therefore $L^p\{X\}$ is reflexive for every $p > 1$.*

PROOF: Consider a function $f^{**}(t) \in L^p\{X^{**}\}$ and a function $\Phi(t) \in V_0^q\{X^*\}$. Then we can form the integral

$$\int_0^{2\pi} f^{**}(t)d\Phi(t), \tag{7.1}$$

defined as an extension of an integral of type (2.3) in the same way that (2.6) was derived from (2.1). The above integral defines a linear functional on the space $V_0^q\{X^*\}$, which is equivalent to the conjugate space $[L^p\{X\}]^*$. Since $L^p\{X\}$ is reflexive every functional on $[L^p\{X\}]^*$ has the form

$$\int_0^{2\pi} d\Phi(t)f(t), \tag{7.2}$$

where $\Phi(t) \in V_0^q\{X^*\}$ and $f(t) \in L^p\{X\}$, the latter function being determined by the given functional. Consequently with $f^{**}(t)$ is associated a certain $f(t) \in L^p\{X\}$ such that (7.1) and (7.2) are equal for all $\Phi(t) \in V_0^q\{X^*\}$. If in particular we take for f^{**} a constant function $f^{**}(t) = x^{**}$, where $x^{**} \in X^{**}$ then for a suitable $f(t) \in L^p\{X\}$

$$\int_0^{2\pi} f^{**}(t)d\Phi(t) = x^{**}[\Phi(2\pi)] = \int_0^{2\pi} d\Phi(t)f(t). \tag{7.3}$$

If we choose, for a given n, and $\varphi \in X^*$

$$\Phi(t) = \int_0^t e^{-ins}\varphi\, ds,$$

then $\Phi(0) = \Phi(2\pi) = 0$ if $n \neq 0$. Therefore

$$\int_0^{2\pi} d\Phi(t)f(t) = \varphi\left(\int_0^{2\varphi} e^{-int}f(t)\, dt\right) = 0,$$

and so, for the Fourier coefficients of $f(t)$, $f_n = 0$ if $n \neq 0$. By Theorem 6.3 the relation (7.3) becomes

$$x^{**}[\Phi(2\pi)] = 2\pi\Phi_0(f_0),$$

where

$$\Phi_0 = \frac{1}{2\pi}\int_0^{2\pi} d\Phi(t) = \frac{1}{2\pi}\Phi(2\pi),$$

$$f_0 = \frac{1}{2\pi}\int_0^{2\pi} f(t)dt.$$

Therefore, writing $\varphi = \Phi(2\pi)$, we get

$$x^{**}(\varphi) = \varphi(f_0).$$

This expresses precisely the fact that X is reflexive, since φ may be arbitrary.

We next shall prove that if $F(t) \in V^p\{X\}$ it is the indefinite integral of a function in $L^p\{X\}$. If $\varphi(t) \in L^q\{X^*\}$ we form the integral

$$\int_0^{2\pi} \varphi(t)\, dF(t) \tag{7.4}$$

[in the same way that (7.1) was defined]. It defines a linear functional on $L^q\{X^*\}$. Now $L^q\{X^*\}$ is equivalent to a closed subspace of $V_0^q\{X^*\}$, which is in turn equivalent to $[L^p\{X\}]^*$. Since $L^p\{X\}$ is reflexive all the functionals on $V_0^q\{X^*\}$ have the form (7.2). It then follows by Hahn-Banach extension theorem[11] that the integral (7.4) must coincide with an integral of the form (7.2) whenever $\Phi(t)$ has the form

$$\Phi(t) = \int_0^t \varphi(s)\, ds.$$

In other words

$$\int_0^{2\pi} \varphi(t)dG(t) = 0, \qquad \varphi(t) \in L^q\{X^*\},$$

where

$$G(t) = F(t) - \int_0^t f(t)\, dt.$$

From this it follows that $G(t)$, which is continuous, is necessarily a constant Thus $F(t)$ is an integral of $f(t)$.

The space $L^q\{X^*\}$ is reflexive also, for its conjugate space is $V_0^p\{X^{**}\}$, and so, by the reflexiveness of X (7.4) is the form of the general linear functional on $L^q\{X^*\}$. But since, by the above, $V_0^p\{X\}$ is equivalent to $L^p\{X\}$, (7.4) takes the form

$$\int_0^{2\pi} \varphi(t)f(t)\, dt, \qquad f(t) \in L^p\{x\},$$

and so the conjugate space of $L^q\{X^*\}$ is $L^p\{X\}$, which is reflexive. By Theorem 5.1 $L^q\{X^*\}$ must itself be reflexive. Hence, by the same reasoning used above $V^q\{X^*\}$ consists entirely of indefinite integrals of functions of $L^q\{X^*\}$.

Now the property of $V^q\{X^*\}$ just stated is equivalent to the assertion that X^* satisfies condition (D). For it is known[35] that condition (D) is satisfied if every function in $V^\infty\{X^*\}$ has a derivative a.e. and certainly $V^\infty\{X^*\}$ is contained in $V^q\{X^*\}$. Similar remarks apply to X. Thus the theorem is completely proved.

8. Linear Operations

It is our purpose in this section to study the form of general linear operations[36] on an arbitrary space X to one of the function spaces M, C, L^p. We begin with L^p $(p > 1)$.

THEOREM 8.1. *If T is a linear operation on X to L^p there exists a function $F(t) \in C\{X^*\}$ such that for each $x \in X$, $F(t)x$ is the indefinite integral of a function in L^p, and T is representable in the form*

$$Tx = \frac{d}{dt}[F(t)x]. \tag{8.1}$$

Conversely, for each $F(t) \in C\{X^*\}$ such that $F(t)x$ is the indefinite integral of a function in L^p, and

$$\left(\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{d}{dt}[F(t)x]\right|^p\frac{d}{dt}\right)^{1/p} \leqq A\,\|x\| \tag{8.2}$$

for some $A > 0$ and all $x \in X$, (8.1) defines a linear operation on X to L^p.

PROOF. If T is given, denote by $f_x(t)$ the element of L^p corresponding to x under the operation. Then

$$\left(\int_0^{2\pi}|f_x(t)|^p\,dt\right)^{1/p} \leqq \|T\|\,\|x\|.$$

Form the Fourier coefficients of $f_x(t)$; they depend on x:

$$\varphi_n(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{-int}f_x(t)\,dt. \tag{8.3}$$

$\varphi_n(x)$ is linear in x, for by Hölder's inequality $|\varphi_n(x)| \leqq (2\pi)^{-1/p}\|T\|\,\|x\|$. Now $f_x(t) \in L^p$. If $1 < p \leqq 2$ we have, by the Hausdorff-Young theorem[34]

$$\left(\sum_{-\infty}^{\infty}{}_n\,|\varphi_n(x)|^q\right)^{1/q} \leqq \left(\frac{1}{2\pi}\int_0^{2\pi}|f_x(t)|^p\,dt\right)^{1/p} \leqq (2\pi)^{-1/p}\|T\|\,\|x\|,$$

[35] See §4 of the paper of Dunford and Morse referred to in footnote 4).

[36] Compare with Nelson Dunford, *Integration and Linear Operations*, Transactions of the American Mathematical Society, vol. 40 (1936), pp. 474–494.

while if $p > 2$, $f_x(t) \in L^2$ and

$$\left(\sum_{-\infty}^{\infty}{}_n |\varphi_n(x)|^2\right)^{\frac{1}{2}} = \left(\frac{1}{2\pi}\int_0^{2\pi} |f_x(t)|^2\,dt\right)^{\frac{1}{2}}$$
$$\leqq (2\pi)^{-1/p}\left(\int_0^{2\pi} |f_x(t)|^p\,dt\right)^{1/p} \leqq (2\pi)^{-1/p}\,\|T\|\,\|x\|.$$

In either case it may be proved, much as in the proof of Theorem 6.5, that the series

$$F(t) = \sum_{-\infty}^{\infty}{}'_\nu \frac{e^{i\nu t}-1}{i\nu}\varphi_\nu + t\varphi_0. \tag{8.4}$$

defines a continuous function on $(0, 2\pi)$ to X^*. But also

$$\sum_{-\infty}^{\infty}{}'_\nu \frac{e^{i\nu t}-1}{i\nu}\varphi_\nu(x) = \int_0^t [f_x(t) - \varphi_0(x)]\,dt$$

and therefore

$$F(t)x = \int_0^t f_x(t)\,dt,$$

whence (8.1) is satisfied.

The converse is trivial.

The theorem pertaining to operations on X to M is similar, but we can assert much more about the nature of the function F.

THEOREM 8.2. *If T is a linear operation on X to M there exists a function $F(t) \in V_0^\infty\{X^*\}$ such that T is representable in the form*

$$Tx = \frac{d}{dt}[F(t)x]. \tag{8.5}$$

The norm of T is given by $\|T\| = N(F)$. Conversely to each $F(t) \in V_0^\infty\{X^*\}$ corresponds a linear operation on X to M, defined by (8.5).

PROOF: As before we write $Tx = f_x(t)$ and consider the Fourier series of $f_x(t)$. We have $|\varphi_n(x)| \leqq \sup_t^0 |f_x(t)| \leqq \|T\|\,\|x\|$, so that the Fourier coefficients are linear functionals of x. We can define $F(t)$ by (8.4) and show that it satisfies (8.5), just as before. Now if t, $t+h$ are in $(0, 2\pi)$

$$|F(t+h)x - F(t)x| = \left|\int_t^{t+h} f_x(t)\,dt\right| \leqq |h|\,\|T\|\,\|x\|.$$

Therefore

$$\|F(t+h) - F(t)\| \leqq |h|\,\|T\|.$$

Consequently, $F(t) \in V_0^\infty\{X^*\}$ and $N(F) \leqq \|T\|$. On the other hand $|f_x(t)| \leqq N(F)\,\|x\|$ a.e. so that $\|T\| = N(F)$. The converse part of the theorem is trivial.

COROLLARY. *If X is separable and T is a linear operation on X to M there exists a function $f(t)$ on $(0, 2\pi)$ to X^*, defined a.e. such that T is given by*

$$Tx = f(t)x. \tag{8.6}$$

This follows at once from the lemma following Theorem 4.3. We also remark that T is representable in the form (8.6) if X^* satisfies condition (D), and in this case $f(t) \in M\{X^*\}$, $\| T \| = \sup_{0 \leq t \leq 2\pi}^{0} \| f(t) \|$.

THEOREM 8.3. *The most general linear operation on X to C, when X is separable, has the form*

$$Tx = f(t)x,$$

where $f(t)$ is a function on $(0, 2\pi)$ to X^* with the properties

a) *for each $x \in X$, $f(t)x$ is continuous $0 \leqq t \leqq 2\pi$,*

b) $| f(t)x | \leqq \| T \| \| x \|$.

PROOF: If $Tx = f_x(t)$ we know from Theorem 8.2, since C is a subspace of M, that there exists a function $F(t) \in V_0^\infty\{X^*\}$ such that

$$F(t)x = \int_0^t f_x(t)\, dt.$$

By the corollary there exists a function $f_0(t)$ with values in X^*, defined a.e., such that $f_0(t)x = f_x(t)$ a.e. Our problem is to show that $f_0(t)$ can be defined at *all* points of the interval $(0, 2\pi)$ in such a way that $f_0(t)x = f_x(t)$ throughout. Since $f_x(t)$ is continuous when x is fixed there is no difficulty about this.

The same conclusion can be drawn if X^* satisfies condition (D), whether X is separable or not.

THEOREM 8.4. *The most general linear operation on X to L is representable in the form*

$$Tx = \frac{d^2}{dt^2}[G(t)x],$$

where $G(r) \in C\{X^*\}$ and for each $x \in X$ $G(t)x$ is the indefinite integral of an absolutely continuous function.

PROOF: If $Tx = f_x(t)$ and $\{\varphi_n(x)\}$ are the Fourier coefficients of $f_x(t)$ it is obvious that $\varphi_n(x)$ is a linear functional of x such that $| \varphi_n(x)| \leqq 1/2\pi \| T \| \| x \|$. Then we put

$$F_x(t) = \int_0^t f_x(s)\, ds = \sum_{-\infty}^{\infty}{}' \frac{e^{i\nu t} - 1}{i\nu} \varphi_\nu(x) + t\varphi_0(x)$$

and note that $F_x(t)$ is absolutely continuous. If

$$H_x(t) = \sum_{-\infty}^{\infty}{}' \frac{e^{i\nu t}}{i\nu} \varphi_\nu(x) = F_x(t) + \sum_{-\infty}^{\infty}{}' \frac{\varphi_\nu(x)}{i\nu} - t\varphi_0(x),$$

then

$$\int_0^t H_x(t)\,dt = \sum_{-\infty}^{\infty}{}' \frac{e^{i\nu t} - 1}{(i\nu)^2}\,\varphi_\nu(x).$$

But the series

$$g(t) = \sum_{-\infty}^{\infty}{}' \frac{e^{i\nu t} - 1}{(i\nu)^2}\,\varphi_\nu$$

converges uniformly and defines a continuous function $g(t)$ with values in X^*. If $G(t) = g(t) + \frac{1}{2}t^2\varphi_0$ then clearly $d^2/dt^2\,[G(t)x] = f_x(t)$, which is the desired result.

PRINCETON UNIVERSITY.

YALE UNIVERSITY
FEb 25 1938
LIBRARY

ANNALS OF MATHEMATICS

(FOUNDED BY ORMOND STONE)

EDITED BY

S. LEFSCHETZ J. VON NEUMANN

F. BOHNENBLUST

WITH THE COÖPERATION OF THE

DEPARTMENT OF MATHEMATICS OF PRINCETON UNIVERSITY

AND

THE SCHOOL OF MATHEMATICS OF THE INSTITUTE FOR ADVANCED STUDY

AND

A. A. Albert	Oystein Ore	H. S. Vandiver
H. Bateman	J. F. Ritt	A. Pell-Wheeler
G. D. Birkhoff	M. H. Stone	Norbert Wiener
E. Hille	J. D. Tamarkin	O. Zariski

SECOND SERIES, VOL. 39, NO. 1

JANUARY, 1938

PUBLISHED QUARTERLY

AT

MOUNT ROYAL AND GUILFORD AVENUES

BALTIMORE, MD.

BY THE

PRINCETON UNIVERSITY PRESS

PRINCETON, N. J.

COPYRIGHT, 1938, BY PRINCETON UNIVERSITY PRESS

ANNALS OF MATHEMATICS

Subscription price, five dollars a volume (four numbers) in advance. Single copies one dollar and fifty cents. An arrangement has been made with the Mathematical Association of America by which an individual member of the Association may subscribe to the Annals of Mathematics at one-half the regular price. Subscriptions, orders for back numbers and changes of address should be sent to Mount Royal and Guilford Avenues, Baltimore, Maryland, or preferably to the Princeton University Press, Princeton, New Jersey.

Manuscripts and all editorial correspondence should be addressed to the Annals of Mathematics, Fine Hall, Princeton, New Jersey. Manuscripts should be typewritten double-spaced. Footnotes should be numbered consecutively and typed (double-spaced) on a separate sheet; they will not be inserted in the text until page proofs are made.

Figures and diagrams should be drawn in India ink on separate sheets of paper and twice the size they are to be printed. The lettering is to be omitted from these drawings, but a sketch with complete lettering should be appended.

Authors ordinarily receive galley proofs only. Stylistic changes must be avoided in proof; excessive corrections will be charged to the authors. Authors receive gratis, postage prepaid, one hundred reprints of each article; joint authors receive fifty reprints each.

SUGGESTIONS TO AUTHORS

Footnotes should be reduced to a minimum or replaced whenever possible by a bibliography at the end of the paper; formulae in footnotes should be avoided. Authors are requested to keep in mind the typographical difficulties of complicated formulae. For example, "exp" may be used to avoid complicated exponents and a power to replace the troublesome radical sign. Symbols set one on top of another present a typographical difficulty which in most cases can easily be avoided by a stepwise arrangement.

Princeton University Press. Agent for countries outside the United States and Canada: Nordemann Publishing Co. N. V., O. Z. 243 Voorburgwal, Amsterdam.

Entered as second-class matter at the Post Office at Baltimore, Maryland, under the Act of March 3, 1879.

Made in United States of America

Copies of the following memoirs can be obtained by addressing THE PRINCETON UNIVERSITY PRESS, Princeton, N. J.:

Fermat's last theorem and the origin and nature of the theory of algebraic numbers. By L. E. DICKSON. 27 pages. Price 35 cents.

An introduction to the theory of elliptic functions. By GOSTA MITTAG-LEFFLER. Authorized translation by EINAR HILLE. 81 pages. Price 90 cents.

Differential equations from the ground standpoint. By L. E. DICKSON. 92 pages. Price $1.00.

On various conceptions of correlation. By F. M. WEIDA. 37 pages. Price 75 cents.

Contact transformations. By L. P. EISENHART. 40 pages. Price 80 cents.

A survey of the theory of small samples. By P. R. RIDER. 52 pages. Price $1.00.

Tauberian theorems. By NORBERT WIENER. 100 pages. Price $1.00.

Untersuchungen über Gestalt und Lage abgeschlossener Mengen beliebiger Dimension. By PAUL ALEXANDROFF. 86 pages. Price $1.00.

Plateau's problem and Dirichlet's principle. By RICHARD COURANT. 46 pages. Price 90 cents.

TABLE OF CONTENTS

On the Set of Positive Functions in L_2. By BÉLA DE SZ. NAGY 1
A Theorem on Analytic Continuation of Functions in Several Variables. By S. BOCHNER 14
Extensions of Linear Functionals, with Applications to Limits, Integrals, Measures, and Densities. By R. P. AGNEW AND A. P. MORSE 20
Green's Formulas for Analytic Functions. By HILLEL PORITSKY 31
The Asymptotic Waring Problem for Homogeneous Polynomial Summands. By H. CHATLAND 49
On A Certain Class of Non-Linear Expansions of an Arbitrary Analytic Function. By GERTRUDE S. KETCHUM AND P. W. KETCHUM 58
The Gravitational Equations and the Problem of Motion. By A. EINSTEIN, L. INFELD, AND B. HOFFMANN 65
Note on the Preceding Paper: The Two Body Problem in General Relativity. By H. P. Robertson 101
Finite Approximations to Lie Groups. By A. M. TURING 105
Lattices and Topological Spaces. By HENRY WALLMAN 112
Transformations of Finite Period. By P. A. SMITH 127
On Projective Normal Coördinates. By SHIING-SHEN CHERN 165
Relations Between Ranks of a General Matrix. By RUFUS OLDENBURGER 172
The Canonical Form of a Matrix. By J. H. M. WEDDERBURN 178
Simple Lie Algebras of Type A. By N. JACOBSON 181
On the Topology of Real Plane Algebraic Curves. By I. PETROWSKY 189
Arithmetical Properties of Sequences in Rings. By MORGAN WARD 210
Group Rings and Extensions. I. By MARSHALL HALL 220
On Finite Systems of Linear Differential Equations of Infinite Order with Constant Coefficients. By GAINES B. LANG 235
Irregularity in Complexes. By S. WYLIE 247

YALE UNIVERSITY
JUN 11 1938
LIBRARY

ANNALS OF MATHEMATICS

(FOUNDED BY ORMOND STONE)

EDITED BY

S. LEFSCHETZ J. VON NEUMANN

F. BOHNENBLUST

WITH THE COÖPERATION OF THE

DEPARTMENT OF MATHEMATICS OF PRINCETON UNIVERSITY

AND

THE SCHOOL OF MATHEMATICS OF THE INSTITUTE FOR ADVANCED STUDY

AND

A. A. ALBERT	OYSTEIN ORE	H. S. VANDIVER
H. BATEMAN	J. F. RITT	A. PELL-WHEELER
G. D. BIRKHOFF	M. H. STONE	NORBERT WIENER
E. HILLE	J. D. TAMARKIN	O. ZARISKI

SECOND SERIES, VOL. 39, NO. 2

APRIL, 1938

PUBLISHED QUARTERLY

AT

MOUNT ROYAL AND GUILFORD AVENUES

BALTIMORE, MD.

BY THE

PRINCETON UNIVERSITY PRESS

PRINCETON, N. J.

COPYRIGHT, 1938, BY PRINCETON UNIVERSITY PRESS

ANNALS OF MATHEMATICS

Subscription price, five dollars a volume (four numbers) in advance. Single copies one dollar and fifty cents. An arrangement has been made with the Mathematical Association of America by which an individual member of the Association may subscribe to the Annals of Mathematics at one-half the regular price. Subscriptions, orders for back numbers and changes of address should be sent to Mount Royal and Guilford Avenues, Baltimore, Maryland, or preferably to the Princeton University Press, Princeton, New Jersey.

Manuscripts and all editorial correspondence should be addressed to the Annals of Mathematics, Fine Hall, Princeton, New Jersey. Manuscripts should be typewritten double-spaced. Footnotes should be numbered consecutively and typed (double-spaced) on a separate sheet; they will not be inserted in the text until page proofs are made.

Figures and diagrams should be drawn in India ink on separate sheets of paper and twice the size they are to be printed. The lettering is to be omitted from these drawings, but a sketch with complete lettering should be appended.

Authors ordinarily receive galley proofs only. Stylistic changes must be avoided in proof; excessive corrections will be charged to the authors. Authors receive gratis, postage prepaid, one hundred reprints of each article; joint authors receive fifty reprints each.

SUGGESTIONS TO AUTHORS

Footnotes should be reduced to a minimum or replaced whenever possible by a bibliography at the end of the paper; formulae in footnotes should be avoided. Authors are requested to keep in mind the typographical difficulties of complicated formulae. For example, "exp" may be used to avoid complicated exponents and a power to replace the troublesome radical sign. Symbols set one on top of another present a typographical difficulty which in most cases can easily be avoided by a stepwise arrangement.

Princeton University Press. Agent for countries outside the United States and Canada: Nordemann Publishing Co. N. V., O. Z. 243 Voorburgwal, Amsterdam.

Entered as second-class matter at the Post Office at Baltimore, Maryland, under the Act of March 3, 1879.

Made in United States of America

Copies of the following memoirs can be obtained by addressing THE PRINCETON UNIVERSITY PRESS, Princeton, N. J.:

Fermat's last theorem and the origin and nature of the theory of algebraic numbers. By L. E. DICKSON. 27 pages. Price 35 cents.

An introduction to the theory of elliptic functions. By GOSTA MITTAG-LEFFLER. Authorized translation by EINAR HILLE. 81 pages. Price 90 cents.

Differential equations from the ground standpoint. By L. E. DICKSON. 92 pages. Price $1.00.

On various conceptions of correlation. By F. M. WEIDA. 37 pages. Price 75 cents.

Contact transformations. By L. P. EISENHART. 40 pages. Price 80 cents.

A survey of the theory of small samples. By P. R. RIDER. 52 pages. Price $1.00.

Tauberian theorems. By NORBERT WIENER. 100 pages. Price $1.00.

Untersuchungen über Gestalt und Lage abgeschlossener Mengen beliebiger Dimension. By PAUL ALEXANDROFF. 86 pages. Price $1.00.

Plateau's problem and Dirichlet's principle. By RICHARD COURANT. 46 pages. Price 90 cents.

TABLE OF CONTENTS

Über einen Faber'schen Satz. Von Paul Erdös und Géza Grünwald 257
Orthogonal Polynomials in Two Complex Variables. By Dunham Jackson 262
Representations for Entire Functions of Exponential Type. By R. P. Boas, Jr. 269
Closure Theorems For Translations. By R. P. Boas, Jr. and S. Bochner 287
Convex Regions and Projections in Minkowski Spaces. By F. Bohnenblust 301
The Plateau Problem For Minimal Surfaces Which Are Relative Minima. By Max Shiffman 309
Fields of Parallel Vectors in Riemannian Space. By Luther Pfahler Eisenhart 316
Über die Dichte der Summe von Mengen positiver ganzer Zahlen. I Von Alfred Brauer 322
Normal Extensions of Quartic Fields with the Symmetric Group. By D. M. Dribin 341
On Congruences Involving Bernoulli Numbers and the Quotients of Fermat and Wilson. By Emma Lehmer 350
Some Studies on Regular Representations, Induced Representations and Modular Representations. By Tadasi Nakayama 361
The Uniqueness of The Power Series Representation of Certain Fields with Valuations. By Saunders Mac Lane 370
On Chains of Topological Spaces. By S. Lefschetz 383
On Products In a Complex. By Hassler Whitney 397
On the Fourier Coefficients of Certain Modular Forms of Positive Dimension. By Hans Rademacher and Herbert S. Zuckerman 433
Arc Length in Metric and Finsler Manifolds. By Sumner Byron Myers 463
The Linear Homogeneous Group. By Joel Brenner 472
Quadratic Null Forms Over a Function Field. By A. Adrian Albert 494

YALE UNIVERSITY
AUG 18 1938
LIBRARY

ANNALS OF MATHEMATICS

(FOUNDED BY ORMOND STONE)

EDITED BY

S. LEFSCHETZ J. VON NEUMANN

F. BOHNENBLUST

WITH THE COÖPERATION OF THE

DEPARTMENT OF MATHEMATICS OF PRINCETON UNIVERSITY

AND

THE SCHOOL OF MATHEMATICS OF THE INSTITUTE FOR ADVANCED STUDY

AND

A. A. ALBERT	OYSTEIN ORE	H. S. VANDIVER
H. BATEMAN	J. F. RITT	A. PELL-WHEELER
G. D. BIRKHOFF	M. H. STONE	NORBERT WIENER
E. HILLE	J. D. TAMARKIN	O. ZARISKI

SECOND SERIES, VOL. 39, NO. 3

JULY, 1938

PUBLISHED QUARTERLY

AT

MOUNT ROYAL AND GUILFORD AVENUES

BALTIMORE, MD.

BY THE

PRINCETON UNIVERSITY PRESS

PRINCETON, N. J.

COPYRIGHT, 1938, BY PRINCETON UNIVERSITY PRESS

ANNALS OF MATHEMATICS

Subscription price, five dollars a volume (four numbers) in advance. Single copies one dollar and fifty cents. An arrangement has been made with the Mathematical Association of America by which an individual member of the Association may subscribe to the Annals of Mathematics at one-half the regular price. Subscriptions, orders for back numbers, and changes of address should be sent

from the United States and Canada to Mount Royal and Guilford Avenues, Baltimore, Maryland, or preferably to the Princeton University Press, Princeton, N. J.

from all other countries, to Nordemann Publishing Company, N. V., 243 O. Z. Voorburgwal, Amsterdam C., Holland.

Manuscripts and all editorial correspondence should be addressed to the Annals of Mathematics, Fine Hall, Princeton, New Jersey. Manuscripts should be typewritten double-spaced. Footnotes should be numbered consecutively and typed (double-spaced) on a separate sheet; they will not be inserted in the text until page proofs are made.

Figures and diagrams should be drawn in India ink on separate sheets of paper and twice the size they are to be printed. The lettering is to be omitted from these drawings, but a sketch with complete lettering should be appended.

Authors ordinarily receive galley proofs only. Stylistic changes must be avoided in proof; excessive corrections will be charged to the authors. Authors receive gratis, postage prepaid, one hundred reprints of each article; joint authors receive fifty reprints each.

SUGGESTIONS TO AUTHORS

Footnotes should be reduced to a minimum or replaced whenever possible by a bibliography at the end of the paper; formulae in footnotes should be avoided. Authors are requested to keep in mind the typographical difficulties of complicated formulae. For example, "exp" may be used to avoid complicated exponents and a power to replace the troublesome radical sign. Symbols set one on top of another present a typographical difficulty which in most cases can easily be avoided by a stepwise arrangement.

Princeton University Press. Agent for countries outside the United States and Canada: Nordemann Publishing Co. N. V., O. Z. 243 Voorburgwal, Amsterdam.

Entered as second-class matter at the Post Office at Baltimore, Maryland, under the Act of March 3, 1879.

Made in United States of America

Copies of the following memoirs can be obtained by addressing

for *the United States, Canada,* and *South America,* The Nordemann Publishing Company, Inc., 215 Fourth Ave., New York, N. Y.

for all other countries, The Nordemann Publishing Company, N. V., 243 O. Z. Voorburgwal, Amsterdam, Holland.

Fermat's last theorem and the origin and nature of the theory of algebraic numbers. By L. E. DICKSON. 27 pages. Price 35 cents.

An introduction to the theory of elliptic functions. By GOSTA MITTAG-LEFFLER. Authorized translation by EINAR HILLE. 81 pages. Price 90 cents.

Differential equations from the ground standpoint. By L. E. DICKSON. 92 pages. Price $1.00.

On various conceptions of correlation. By F. M. WEIDA. 37 pages. Price 75 cents.

Contact transformations. By L. P. EISENHART. 40 pages. Price 80 cents.

A survey of the theory of small samples. By P. R. RIDER. 52 pages. Price $1.00.

Tauberian theorems. By NORBERT WIENER. 100 pages. Price $1.00.

Untersuchungen über Gestalt und Lage abgeschlossener Mengen beliebiger Dimension. By PAUL ALEXANDROFF. 86 pages. Price $1.00.

Plateau's problem and Dirichlet's principle. By RICHARD COURANT. 46 pages. Price 90 cents.

TABLE OF CONTENTS

Contact Transformations. By M. S. KNEBELMAN 507
Meromorphic Curves. By HERMANN AND JOACHIM WEYL 516
The Iterated Exponential Integers. By E. T. BELL 539
Structure Residuation. By MORGAN WARD 558
On One Parameter Groups of Linear Transformations. By NELSON DUNFORD 569
Linear Operations Which Depend Analytically on a Parameter. By A. E. TAYLOR 574
Arithmetic and Ideal Theory of Commutative Semigroups. By A. H. CLIFFORD 594
Periodic Transformations of Complexes. By M. RICHARDSON and P. A. SMITH 611
On the Regular Representations of Algebras. By C. NESBITT 634
Note on Symmetric Algebras. By TADASI NAKAYAMA and CECIL NESBITT .. 659
On Cyclic Algebras. By A. ADRIAN ALBERT 669
On a Generalization of Kaluza's Theory of Electricity. By A. EINSTEIN and P. BERGMANN 683

R.R.
9

YALE UNIVERSITY
NOV 8 1938
LIBRARY

ANNALS OF MATHEMATICS

(FOUNDED BY ORMOND STONE)

EDITED BY

S. LEFSCHETZ J. VON NEUMANN

F. BOHNENBLUST

WITH THE COÖPERATION OF THE

DEPARTMENT OF MATHEMATICS OF PRINCETON UNIVERSITY

AND

THE SCHOOL OF MATHEMATICS OF THE INSTITUTE FOR ADVANCED STUDY

AND

A. A. ALBERT
H. BATEMAN
G. D. BIRKHOFF
E. HILLE
OYSTEIN ORE
J. F. RITT
M. H. STONE
J. D. TAMARKIN
H. S. VANDIVER
A. PELL-WHEELER
NORBERT WIENER
O. ZARISKI

SECOND SERIES, VOL. 39, NO. 4

OCTOBER, 1938

PUBLISHED QUARTERLY
AT
MOUNT ROYAL AND GUILFORD AVENUES
BALTIMORE, MD.
BY THE
PRINCETON UNIVERSITY PRESS
PRINCETON, N. J.

COPYRIGHT, 1938, BY PRINCETON UNIVERSITY PRESS

ANNALS OF MATHEMATICS

Subscription price, five dollars a volume (four numbers) in advance. Single copies one dollar and fifty cents. An arrangement has been made with the Mathematical Association of America by which an individual member of the Association may subscribe to the Annals of Mathematics at one-half the regular price. Subscriptions, orders for back numbers, and changes of address should be sent

from the United States and Canada to Mount Royal and Guilford Avenues, Baltimore, Maryland, or preferably to the Princeton University Press, Princeton, N. J.

from all other countries, to Nordemann Publishing Company, N. V., 243 O. Z. Voorburgwal, Amsterdam C., Holland.

Manuscripts and all editorial correspondence should be addressed to the Annals of Mathematics, Fine Hall, Princeton, New Jersey. Manuscripts should be typewritten double-spaced. Footnotes should be numbered consecutively and typed (double-spaced) on a separate sheet; they will not be inserted in the text until page proofs are made.

Figures and diagrams should be drawn in India ink on separate sheets of paper and twice the size they are to be printed. The lettering is to be omitted from these drawings, but a sketch with complete lettering should be appended.

Authors ordinarily receive galley proofs only. Stylistic changes must be avoided in proof; excessive corrections will be charged to the authors. Authors receive gratis, postage prepaid, one hundred reprints of each article; joint authors receive fifty reprints each.

SUGGESTIONS TO AUTHORS

Footnotes should be reduced to a minimum or replaced whenever possible by a bibliography at the end of the paper; formulae in footnotes should be avoided. Authors are requested to keep in mind the typographical difficulties of complicated formulae. For example, "exp" may be used to avoid complicated exponents and a power to replace the troublesome radical sign. Symbols set one on top of another present a typographical difficulty which in most cases can easily be avoided by a stepwise arrangement.

Princeton University Press. Agent for countries outside the United States and Canada: Nordemann Publishing Co. N. V., O. Z. 243 Voorburgwal, Amsterdam.

Entered as second-class matter at the Post Office at Baltimore, Maryland, under the Act of March 3, 1879.

Made in United States of America

Copies of the following memoirs can be obtained by addressing

for *the United States, Canada,* and *South America,* The Nordemann Publishing Company, Inc., 215 Fourth Ave., New York, N. Y.

for all other countries, The Nordemann Publishing Company, N. V., 243 O. Z. Voorburgwal, Amsterdam, Holland.

Fermat's last theorem and the origin and nature of the theory of algebraic numbers. By L. E. DICKSON. 27 pages. Price 35 cents.

An introduction to the theory of elliptic functions. By GOSTA MITTAG-LEFFLER. Authorized translation by EINAR HILLE. 81 pages. Price 90 cents.

Differential equations from the ground standpoint. By L. E. DICKSON. 92 pages. Price $1.00.

On various conceptions of correlation. By F. M. WEIDA. 37 pages. Price 75 cents.

Contact transformations. By L. P. EISENHART. 40 pages. Price 80 cents.

A survey of the theory of small samples. By P. R. RIDER. 52 pages. Price $1.00.

Tauberian theorems. By NORBERT WIENER. 100 pages. Price $1.00.

Untersuchungen über Gestalt und Lage abgeschlossener Mengen beliebiger Dimension. By PAUL ALEXANDROFF. 86 pages. Price $1.00.

Plateau's problem and Dirichlet's principle. By RICHARD COURANT. 46 pages. Price 90 cents.

TABLE OF CONTENTS

On Interpolation. II. On the Distribution of the Fundamental Points of Lagrange and Hermite Interpolation. By P. Erdös and P. Turán . . 703

Hypersurfaces in a Projective Curved Space. By V. Hlavatý 725

Hypersurfaces of a Space of Constant Curvature. By Aaron Fialkow . . . 762

Non-Unique Solutions of Systems of First Order Ordinary Differential Equations. By O. H. Hamilton 786

An Abstract Form of the Covering Theorems of Topology. By Karl Menger 794

A General Existence Theorem and Some Applications. By A. N. Milgram. 804

Metric Spaces and Completely Monotone Functions. By I. J. Schoenberg 811

A Remarkable Property of the Riemann-Christoffel Tensor in Four Dimensions. By Cornelius Lanczos 842

Monotone Transformations of Two-Dimensional Manifolds. By J. H. Roberts and N. E. Steenrod 851

Conformal Geometry on a Surface. By George Comenetz 863

On the Permutation of the Convergents of a Continued Fraction and Related Convergence Criteria. By John Q. Jordan and Walter Leighton, Jr. 872

A Theory of Connectivity in Terms of Gratings. By J. W. Alexander . . . 883

Linear Functionals on Certain Spaces of Abstractly-Valued Functions. By S. Bochner and A. E. Taylor 913

Index iii